ON THE ENERGY CONTRIBUTION OF EXPLICIT TIME-INTEGRATION IN PROJECTION METHODS

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INTRODUCTION

The accurate and efficient simulation of incompressible fluid flows remains a cornerstone of computational mechanics, with critial applications ranging from turbulent aerodynamics to geophysical flows. At the heart of these simulations lies the numerical solution of the incompressible Navier-Stokes equations, where the primary computational challenge stems from the incompressibility constraint, which couples the velocity and pressure fields.

Originally proposed by Chorin and Temam in the late 1960s [1, 2], projection methods have become the standard approach for time-advancing the incompressible Navier-Stokes equations. By decoupling the velocity and pressure updates, projection methods split the time-step into two distinct stages: a predictor step that solves for an intermediate velocity field, $\mathbf{u}^*$, and a corrector step that enforces the incompressibility constraint by projecting $\mathbf{u}^*$ onto a divergence-free space via the solution of a Pressure Poisson Equation.

While the splitting error and spatial accuracy of projection methods have been extensively documented in the literature [3], the energy conservation properties of the time-integration schemes employed within these methods have received comparatively less attention. Notably, the choice of time-integration scheme can significantly influence the overall energy behavior of the numerical solution, potentially leading to unphysical energy growth or excessive dissipation in long-term turbulent simulations, and therefore require closer examination.

Sanderse [4] analyzed the energy conservation properties of Runge-Kutta methods applied to incompressible flows and determined under which conditions these methods are exactly energy-conserving. However, the analysis did not tackle the specific details on determining the energy contribution of the time-integration scheme when used in conjunction with projection methods. On the other hand, Capuano et al. [5] proposed explicit Runge-Kutta schemes with improved energy-conservation properties for incompressible flows, but their analysis focused on the design of the schemes rather than a general framework for assessing the energy contribution of time-integration methods within projection methods.

This work aims to fill this gap by analyzing the energy contribution of explicit time-integration schemes, specifically multistep and Runge-Kutta methods, when applied within projection methods for incompressible flows. By deriving the energy evolution equations associated with these schemes, we identify the key factors that influence their dissipation properties and provide guidelines for selecting appropriate time-integration methods that minimize unphysical energy behavior in incompressible flow simulations.

TIME-INTEGRATION OF INCOMPRESSIBLE FLOWS WITH PROJECTION METHODS

Starting from the incompressible Navier-Stokes equations in their semi-discrete form after spatial discretization, these read

$$M\mathbf{u} = 0, \tag{1}$$

$$\Omega \frac{d\mathbf{u}}{dt} + C(\mathbf{u})\mathbf{u} = -\Omega G\mathbf{p} + D\mathbf{u} + \Omega\mathbf{f}, \tag{2}$$

where $\mathbf{u}$ is the discrete velocity vector, $\mathbf{p}$ is the discrete pressure vector, M is the discrete divergence operator, G is the discrete gradient operator, $C(\mathbf{u})$ is the discrete convection operator, D is the discrete diffusion operator, Ω is the diagonal matrix with the volumes of the control volumes in the diagonal, and $\mathbf{f}$ is the discrete body force vector.

The lack of a pressure evolution equation transforms the system into a Differential-Algebraic Equation (DAE) system of index 2, and therefore cannot be treated as a set of ordinary differential equations (ODEs). Even though various approaches exist to solve DAEs directly, all involve the numerical solution of a saddle-point problem at each time step, which is computationally expensive [6]. In order to avoid this issue, projection methods decouple the computation of velocity and pressure by introducing a splitting error [1, 2, 3].

Projection methods first compute an intermediate velocity field $\mathbf{u}^*$ by integrating the ODE that results from neglecting the pressure gradient term in Eq.(2). Following [7], we can define a projection operator $P = I - GL^{-1}M$, where L is the discrete Laplacian operator, so that the whole projection method can be written in compact form as

$$\frac{d\mathbf{u}}{dt} = P\left(-\Omega^{-1}C(\mathbf{u})\mathbf{u} + \Omega^{-1}D\mathbf{u} + \mathbf{f}\right) = PF(\mathbf{u}), \tag{3}$$

where $F(\mathbf{u})$ is the right-hand side of the momentum equation without the pressure gradient term. The projection operator P ensures that the velocity field remains divergence-free at each time step. While in practice projection methods do not explicitly compute P, this formulation is useful for the analysis of time-integration schemes for incompressible flows.

The method used to integrate Eq.(3) in time has therefore a strong influence on the overall accuracy and stability of the projection method. While seminal works in turbulence simulations with projection methods used multistep methods such as Adams-Bashforth [8], more recent works have focused on the use of Runge-Kutta (RK) methods due to their favorable stability and accuracy properties [7]. Later works have also explored the use of General Linear Methods (GLM) [9], which provide a general framework to design time-integration schemes with desired properties.

SELECTING A SUITABLE TIME-INTEGRATION STRATEGY

The selection of an appropriate time-integration scheme for incompressible flows using projection methods is crucial to ensure accurate and stable simulations. Several factors must be considered when choosing a time-integration method, including the order of accuracy, stability properties, computational efficiency, and energy conservation characteristics.

While it is generally desirable to use high-order time-integration schemes to minimize temporal discretization errors, the presence of the splitting error in projection methods generally limits the overall accuracy as it becomes the dominant source of error. Therefore, while high-order schemes can still be beneficial, it is essential to balance the order of accuracy with the splitting error to avoid unnecessary computational costs.

The use of high-order schemes is benefitial due to their improved stability properties, which allow the use of larger time steps without compromising the accuracy of the solution. This is particularly important in turbulent flow simulations, where small time steps can lead to prohibitively long simulation times. Methods like `EigenCD` [10] or `AlgEigCD` [11] can be used to estimate the maximum stable time step for a given time-integration scheme and spatial discretization, therefore maximizing the efficiency of the simulation.

For long-term simulations of turbulent flows, it is essential to consider the energy conservation properties of the time-integration scheme. Unphysical energy growth or excessive dissipation can lead to inaccurate results and misrepresent the underlying physics of the flow. Therefore, selecting time-integration methods with favorable energy behavior is crucial to ensure reliable simulations.

Therefore, the selection of a suitable time-integration strategy for incompressible flows using projection methods requires a careful consideration of various factors, including accuracy, stability, efficiency, and energy conservation. By balancing these considerations, it is possible to choose time-integration schemes that provide accurate and stable simulations while minimizing computational costs. While the first three factors have been extensively studied in the literature, energy conservation properties require further analysis, which is the focus of this work.

DISSIPATION ANALYSIS OF EXPLICIT MULTISTEP METHODS

In general, the analysis of the dissipation properties of time-integration schemes for incompressible flows is not straightforward due to the presence of the projection operator P in Eq.(3). Applying an explicit multistep method to Eq.(3) leads to the following update formula for the velocity field

$$\mathbf{u}^{n+1} = \mathbf{u}^n + \Delta t \sum_{i=0}^{k} \beta_i P F(\mathbf{u}^{n-i}), \tag{4}$$

where β_i are the coefficients of the multistep method. Therefore, the dissipation properties of the multistep method are influenced by the interaction between the projection operator P and the right-hand side $F(\mathbf{u})$.

Following the analysis of [4] the energy evolution equation can be obtained by performing the inner product of $\mathbf{u}^{n+1}$, which reads

$$||\mathbf{u}^{n+1}||^2 = ||\mathbf{u}^n||^2 + 2\Delta t \sum_{i=0}^{k} \beta_i (\mathbf{u}^n, PF(\mathbf{u}^{n-i})) + \Delta t^2 \sum_{i,j=0}^{k} \beta_i \beta_j (PF(\mathbf{u}^{n-i}), PF(\mathbf{u}^{n-j})). \tag{5}$$

While the contributions from the third term on the right-hand side of Eq.(5) are strictly coming from the time-integration scheme, the second term includes crossed contributions between the current and previous time steps. Therefore, while the contribution coming from $i = 0$ can be directly associated with the dissipation properties of the spatial schemes used. Generally, spatial schemes following a symmetry-preserving discretization [12, 13] will just introduce dissipation arising from the diffusive term.

On the other hand, contributions with $i > 0$ are influenced by the interaction between the projection operator P and the right-hand side $F(\mathbf{u})$ at previous time steps. This interaction can lead to additional dissipation or even energy growth, depending on the specific multistep method used and the properties of the projection operator.

DISSIPATION ANALYSIS OF EXPLICIT RUNGE-KUTTA METHODS

When applying an explicit Runge-Kutta (RK) method to Eq.(3), the update formula for the velocity field can be written as

$$\mathbf{U}_i = \mathbf{u}^n + \Delta t \sum_{j=1}^{i-1} a_{ij} PF(\mathbf{U}_j), \quad i = 1, \ldots, s, \tag{6}$$

$$\mathbf{u}^{n+1} = \mathbf{u}^n + \Delta t \sum_{i=1}^{s} b_i PF(\mathbf{U}_i), \tag{7}$$

where $\mathbf{U}_i$ are the stage values, a_{ij} and b_i are the coefficients of the RK method, and s is the number of stages.

Similarly to the multistep case, the energy evolution equation can be obtained by performing the inner product of $\mathbf{u}^{n+1}$, which reads

$$||\mathbf{u}^{n+1}||^2 = ||\mathbf{u}^n||^2 + 2\Delta t \sum_{i=1}^{s} b_i (\mathbf{U}_i, P\mathbf{F}_i) + \Delta t^2 \sum_{i,j=1}^{s} m_{ij} (P\mathbf{F}_i, P\mathbf{F}_j), \tag{8}$$

where $\mathbf{F}_i = F(\mathbf{U}_i)$ and $m_{ij} = b_i b_j - b_i a_{ij} - b_j a_{ji}$ is the symplectic matrix associated with the Runge-Kutta scheme, which leads to energy-conserving behavior if it is null [4]. Opposite to multistep methods, the contribution from the second term does not include crossed contribution between different inner stages, and therefore, all contributions in that term can be directly associated with the dissipation properties of the spatial schemes used. The third term, on the other hand, includes crossed terms between different inner stages, which lead to the energy contributions strictly coming from the time-integration scheme.

NUMERICAL RESULTS

To validate the dissipation analysis presented in the previous sections, numerical simulations of different incompressible flow problems will be presented in the conference. The simulations will be performed using various explicit multistep and Runge-Kutta methods within a projection method framework. The energy evolution of the velocity field will be monitored over time to assess the dissipation properties of each time-integration scheme. Moreover, the energy evolution of the different contributions to $F(\mathbf{u})$ will be analyzed to validate the theoretical findings regarding the interaction between the projection operator and the right-hand side of the momentum equation.

The numerical simulations performed will include flow cases with and without the addition of the energy equation, to assess the impact of this addition on the overall energy behavior of the simulations. The results will provide insights into the suitability of different time-integration schemes for incompressible flow simulations using projection methods, and will help to identify best practices for selecting appropriate schemes based on their dissipation properties.

Figure 1 shows the energy budgets obtained for a differentially heated cavity filled with air of aspect ratio 4 at Ra=10^6. The results show the contributions of the different terms in the kinetic energy equation, including the convective, diffusive, buoyancy, and time-integration contributions. The results highlight the importance of accurately capturing the energy contributions from the time-integration scheme to ensure reliable simulations of incompressible flows.

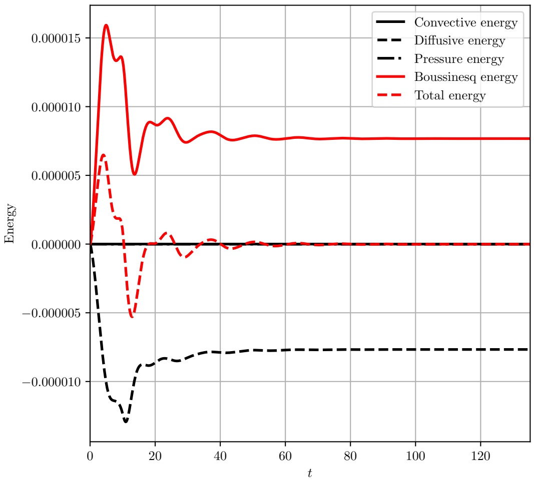

Figure 1: Energy budgets of a differentially heated cavity filled with air of aspect ratio 4 with Ra=10^6.

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