Symmetry-preserving regularization of wall-bounded turbulent flows

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Governing equations

Incompressible Navier-Stokes equations:

$$\begin{aligned} \nabla \cdot u &= 0 \\ \partial_t u + \mathcal{C}(u,u) &= \mathcal{D}(u) - \nabla p \end{aligned}$$

where the **nonlinear convective term** is given by

$$\mathcal{C}(u,\phi) = (u \cdot \nabla)\phi$$

and the linear dissipative term is given by

$$\mathcal{D}(\phi) = \nu \Delta \phi$$

Regularization modeling

As the full energy spectrum cannot be computed, a **dynamically less complex mathematical formulation** is sought. We consider smooth approximations (regularizations) of the nonlinearity,

$$\partial_t u_\epsilon + \widetilde{\mathcal{C}}(u_\epsilon, u_\epsilon) = \mathcal{D}(u_\epsilon) - \nabla p_\epsilon$$

such approximations may fall in the **Large-Eddy Simulation** (LES) concept,

$$\partial_t \bar{u}_\epsilon + \mathcal{C}(\bar{u}_\epsilon, \bar{u}_\epsilon) = \mathcal{D}(\bar{u}_\epsilon) - \nabla \bar{p}_\epsilon + \mathcal{M}_1(\bar{u}_\epsilon, \bar{u}_\epsilon)$$

if the filter is invertible:

$$\mathcal{M}_1(\bar{u}_\epsilon, \bar{u}_\epsilon) = \mathcal{C}(\bar{u}_\epsilon, \bar{u}_\epsilon) - \overline{\widetilde{\mathcal{C}}(u_\epsilon, u_\epsilon)}$$

Symmetry-preserving regularization models

In order to conserve the following inviscid invariants

- Kinetic energy : (u, u)
- Enstrophy (in 2D) : (ω, ω)
- Helicity (in 3D) : (ω, u)

where $(a, b) = \int_\Omega a \cdot b d\Omega$ and $\omega = \nabla \times u$; the **approximate convective operator** must **preserve** the basic **symmetry** properties:

$$(\mathcal{C}(u, v), \; w) = -(\mathcal{C}(u, \; w), v)$$
$$(\mathcal{C}(u, v), \Delta v) = \; (\mathcal{C}(u, \Delta v), v) \qquad \text{in 2D}$$

Symmetry-preserving regularization models

Regularizations of the non-linear convectiver term can be constructed

$$\tilde{\mathcal{C}}(u,v) = \sum_{i,j,k=0}^{1} a_{ijk}\tilde{\mathcal{C}}_{ijk}(u,v)$$

where $\tilde{\mathcal{C}}_{ijk}(u,v) = \varphi_k\left(\mathcal{C}(\varphi_i(u),\varphi_j(v))\right)$ and $\varphi_i(u) = \begin{cases} u\ , & \text{if } i=0 \\ \overline{u}\ , & \text{if } i=1 \end{cases}$

$\overline{(\cdot)}$ is a **self-adjoint** filter that **commutes** with differential operators.

Among all possible combinations we find the regularization proposed by Leray, $\mathcal{C}(\overline{u},u)$: $a_{100}=1$ (with the rest of $a_{ijk}=0$)

$\Longrightarrow$ Eight coefficients a_{ijk} need to be determined.

Symmetry-preserving regularization models

$$\tilde{\mathcal{C}}(u,v) = \sum_{i,j,k=0}^{1} a_{ijk} \tilde{\mathcal{C}}_{ijk}(u,v)$$

$$\sum_{i,j,k=0}^{1} a_{ijk} = 1 \quad \longrightarrow \quad \tilde{\mathcal{C}}(u,v) = \mathcal{C}(u,v) + \mathcal{O}(\epsilon^n) \qquad \text{with } n \geq 2$$

$$\left(\tilde{\mathcal{C}}(u,v), \; w\right) = -\left(\tilde{\mathcal{C}}(u, \; w), v\right) \qquad \longrightarrow \qquad a_{ijk} = a_{ikj}$$

$$\left(\tilde{\mathcal{C}}(u,v), \Delta v\right) = \quad \left(\tilde{\mathcal{C}}(u, \Delta v), v\right) \quad \text{in 2D} \quad \longrightarrow \qquad a_{ijk} = a_{kji}$$

This leads to a family of $\mathcal{O}(\epsilon^2)$-accurate regularizations. Among them[1],

$$\mathcal{C}_2(u,v) = \tilde{\mathcal{C}}_{111}(u,v) = \overline{\mathcal{C}(\overline{u},\overline{v})}$$

[1] Roel Verstappen, **Computers & Fluids**, 37 (7): 887-897, 2008

Symmetry-preserving regularization models

To cancel second-order terms, three additional conditions need to imposed:

$$\sum_{j,k=0}^{1} a_{1jk} = 0 \qquad \sum_{i,k=0}^{1} a_{i1k} = 0 \qquad \sum_{i,j=0}^{1} a_{ij1} = 0$$

$$\boxed{\mathcal{C}_4^{\gamma}(u,v) = \frac{1}{2}\left((\mathcal{C}_4 + \mathcal{C}_6) + \gamma(\mathcal{C}_4 - \mathcal{C}_6)\right)(u,v)}$$

where $\mathcal{C}_4$ and $\mathcal{C}_6$ read

$$\mathcal{C}_4(u,v) = \mathcal{C}(\bar{u},\bar{v}) + \overline{\mathcal{C}(\bar{u},v')} + \overline{\mathcal{C}(u',\bar{v})}$$
$$\mathcal{C}_6(u,v) = \mathcal{C}(\bar{u},\bar{v}) + \mathcal{C}(\bar{u},v') + \mathcal{C}(u',\bar{v}) + \overline{\mathcal{C}(u',v')}$$

Symmetry-preserving regularization models

Taking $\gamma = 1$ we obtain the $\mathcal{C}_4$ approximation[1],

$$\partial_t u_\epsilon + \mathcal{C}_4(u_\epsilon, u_\epsilon) = \mathcal{D}(u_\epsilon) - \nabla p_\epsilon$$

in which the convective term in smoothed according to:

$$\boxed{\mathcal{C}_4(u, v) = \mathcal{C}(\bar{u}, \bar{v}) + \overline{\mathcal{C}(\bar{u}, v')} + \overline{\mathcal{C}(u', \bar{v})}}$$

where $u' = u - \bar{u}$ and $\mathcal{C}_4(u, v) = \mathcal{C}(u, v) + \mathcal{O}(\epsilon^4)$ for **any symmetric filter**.

High-frequencies need to be effectively damped.

But how much?

[1] Roel Verstappen, **Computers & Fluids**, 37 (7): 887-897, 2008

Stopping the vortex-stretching[2]

Taking the curl of momentum equation the vorticity transport equation follows

$$\partial_t \omega + \mathcal{C}(u, \omega) = \mathcal{C}(\omega, u) + \mathcal{D}(\omega)$$

Let us now consider an arbitrary part of the flow domain, Ω, with periodic boundary conditions. Then, taking the L^2 innerproduct with $\omega = \nabla \times u$ leads to the enstrophy equation

$$\frac{1}{2}\frac{d}{dt}(\omega, \omega) = (\omega, \mathcal{C}(\omega, u)) - \nu(\nabla\omega, \nabla\omega)$$

where $(a, b) = \int_\Omega a \cdot b d\Omega$. Unless, the grid is fine enough convection dominates diffusion

$$(\omega, \mathcal{C}(\omega, u)) \; > \; \nu(\nabla\omega, \nabla\omega)$$

[2]F.X. Trias *et al.* **Computers&Fluids**, 39:1815-1831, 2010

Stopping the vortex-stretching

The vortex-stretching term can be expressed in terms of the invariant $r = -1/3tr(S^3)$

$$(\omega, \mathcal{C}(\omega, u)) = 4 \int_\Omega r d\Omega \tag{1}$$

whereas the $L^2(\Omega)$-norm of ω in terms of the invariant $q = -1/2tr(S^2)$

$$(\omega, \omega) = -4 \int_\Omega q d\Omega$$

Then, the diffusive term can be bounded by

$$\nu (\nabla\omega, \nabla\omega) = -\nu (\omega, \Delta\omega) \leq -\nu\lambda_\Delta (\omega, \omega) = 4\nu\lambda_\Delta \int_\Omega q d\Omega \tag{2}$$

where $\lambda_\Delta < 0$ is the largest (smallest in absolute value) non-zero eigenvalue of Laplacian operator Δ on Ω. If we now consider that the domain is a periodic box of volume h, then $\lambda_\Delta = -(\pi/h)^2$.

Stopping the vortex stretching

$\Longrightarrow$ In the present work we **determine the filter width** ϵ from

$$(\omega, \mathcal{C}_4(\omega, u)) \approx f_4(\hat{g}_k(\epsilon))(\omega, \mathcal{C}(\omega, u)) \leq \nu(\nabla\omega, \nabla\omega)$$

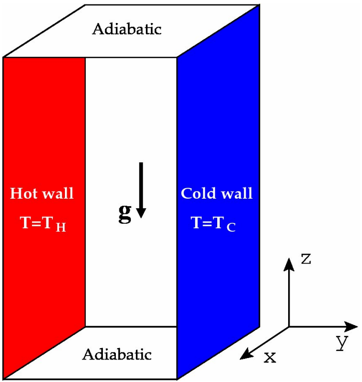
Then, recalling identity (1) and inequality (2), we propose to rewrite the previous inequality in terms of the invariants q and r

$$f_4(\hat{g}_k) = min\left\{\nu\lambda_\Delta \frac{q}{r^+}, 1\right\} \qquad with \quad r^+ = max(r, 0)$$

Notice that $q < 0$ (dissipation) whereas r can be either positive or negative.

- Switches off ($f_4 = 1$) for: laminar ($r \rightarrow 0$), 2D flows ($r = 0$) and for fine enough meshes, $|\nu\lambda_\Delta q/r| \geq 1$.
- Consistent near-wall behavior $r \propto y^3$ and $q \propto y^0$.
- Consistent with the preferential vorticity alignment with the intermediate eigenvector, λ_2 (experimentally observed)

Test-case: Differentially Heated Cavity



Boundary conditions:

- **Isothermal vertical walls**
- **Adiabatic horizontal walls**
- **Periodic** boundary conditions in the spanwise direction

Dimensionless governing numbers:

- $Ra = \beta \Delta T L_z^3 g / (\alpha \nu)$
- $Pr = \nu / \alpha$
- Height aspect ratio $A_z = L_z / L_y$
- Depth aspect ratio $A_x = L_x / L_y$

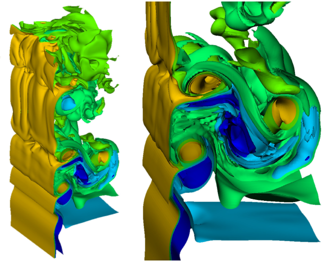
DNS[3,4] results for $Ra = 10^{11}$, $Pr = 0.71$

Some details about **DNS**:

- Mesh size: $128 \times 682 \times 1278$
- $\approx$ 3 months - 256 CPUs
- 4^{th}-order symmetry-preserving scheme
- $A_z = 4$

Complexity of the flow:

- Boundary layers
- Stratified cavity core
- Internal waves
- Recirculation areas



[3] F. X. Trias *et al.* Int. Journal of Heat and Mass Transfer, 53:665-673, 2010

[4] F. X. Trias *et al.* Int. Journal of Heat and Mass Transfer, 53:674-683, 2010

Results for differentially heated cavity at $Ra = 10^{11}$

- Regularization model $\mathcal{C}_4$ is tested.
- Two coarse meshes are considered

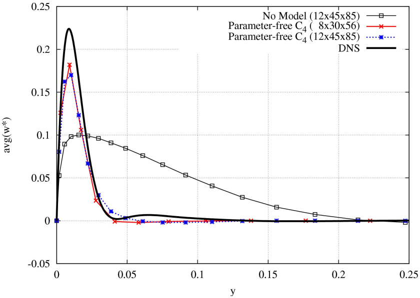
	DNS	RM1	RM2
Nx	128	12	8
Ny	682	45	30
Nz	1278	85	56

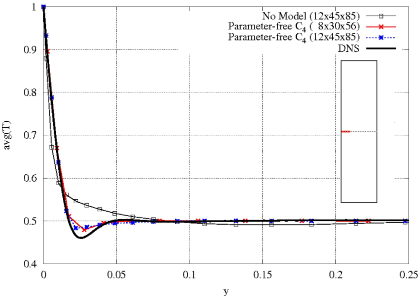
- The **discrete linear filter**[5] is based on polynomial functions of the discrete diffusive operator, D

[5] F.X. Trias and R.W.C.P. Verstappen, **Computers & Fluids**, 40:139-148, 2011

Results for differentially heated cavity at $Ra = 10^{11}$

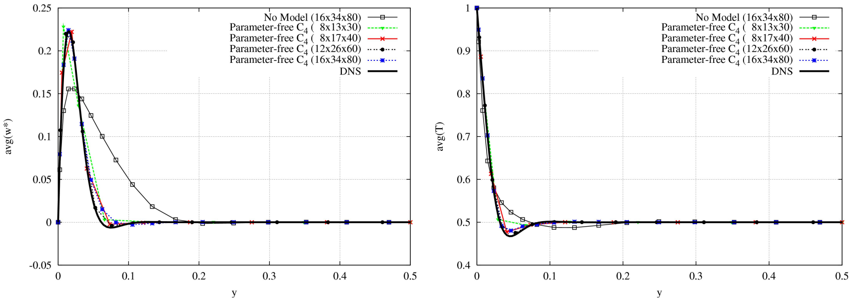
Profiles





Averaged vertical velocity and temperature profiles at the horizontal mid-height plane.

How does the parameter-free $\tilde{\mathcal{C}}_4$ regularization modeling behave for other grids and *Ra*-numbers?

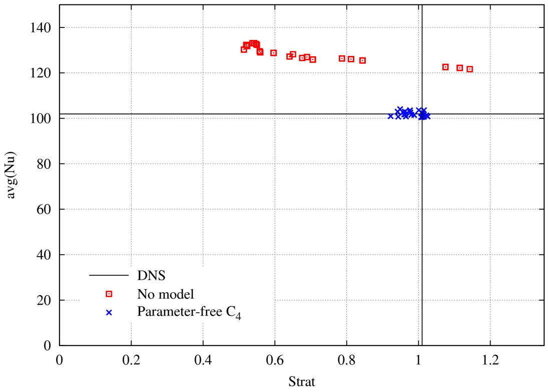


Averaged vertical velocity and temperature profiles at the horizontal mid-height plane at $Ra = 10^{10}$.

Even for a **very coarse** $8 \times 13 \times 30$ grid **reasonable results** are obtained!

$\Longrightarrow$ Results for different grids show the **robustness** of the method.

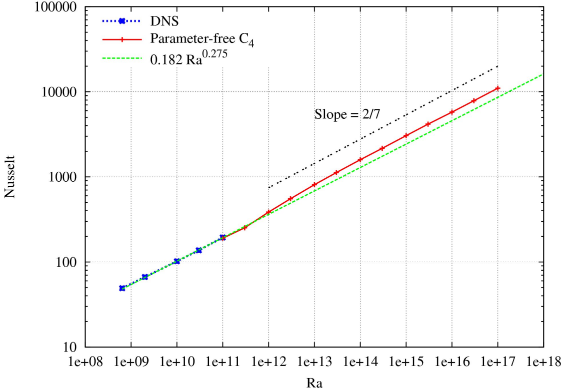
Challenging $\mathcal{C}_4$: mesh independence analysis



The overall Nusselt number and the centerline stratification for 50 randomly generated coarse grids with fixed stretching at $Ra = 10^{10}$.

$8 \leq N_x \leq 16$, $17 \leq N_y \leq 34$, and $40 \leq N_z \leq 80$.

Performance at very high Rayleigh numbers



Meshes have been generated with the criteria of keeping the same number of points in the BL than for $Ra = 10^{10}$.

Conclusions and Future Research

- The results illustrate the potential of $\mathcal{C}_4$ regularization as a **parameter-free turbulence model**.
- **Robustnest**. It preserves the symmetry properties and therefore, the solution cannot blow up even for very coarse meshes.

- Test the performance of other forms of $\mathcal{C}_4^\gamma$ regularization (with $\gamma \neq 1$).
- Add some **additional dissipation** by (approximately) restoring the Galilean invariance.

$$(\partial_t)_4^\gamma u_\epsilon = \partial_t(u_\epsilon - 1/2(1+\gamma)u_\epsilon'') = \mathcal{G}_4^\gamma(\partial_t u_\epsilon),$$

Since $(\mathcal{G}_4^\gamma)^{-1}(\phi) \approx 2\phi - \mathcal{G}_4^\gamma(\phi) + \mathcal{O}(\epsilon^6)$, an energetically almost equivalent set of equations can be derived

$$\partial_t u_\epsilon + \mathcal{C}_4^\gamma(u_\epsilon, u_\epsilon) = \mathcal{D}_4^\gamma u_\epsilon - \nabla p_\epsilon,$$

where $\mathcal{D}_4^\gamma u = \mathcal{D}u + 1/2(1+\gamma)(\mathcal{D}u')'$.

Thank you for you attention

Further reading about $\mathcal{C}_4$ regularization

- Roel Verstappen, *"On restraining the production of small scales of motion in a turbulent channel flow"*, Computers & Fluids, 37 (7): 887-897, 2008
- F. X. Trias *et al.*, *"Parameter-free symmetry-preserving regularization modeling of a turbulent differentially heated cavity"*, Computers & Fluids, 39:1815-1831, 2010.
- F. X. Trias and R.W.C.P. Verstappen, *"On the construction of discrete filters for symmetry-preserving regularization models"*, Computers & Fluids, 40:139-148, 2011.