Second International Conference on Turbulence and Interactions

31 May - 5 June 2009, Sainte-Luce, Martinique, France

Parameter-free symmetry-preserving regularization modelling of turbulent natural convection flows

R.W.C.P. Verstappen [*], ***F.X. Trias*** [*,⋆], ***M. Soria*** [⋆] ***and A. Oliva*** [⋆]

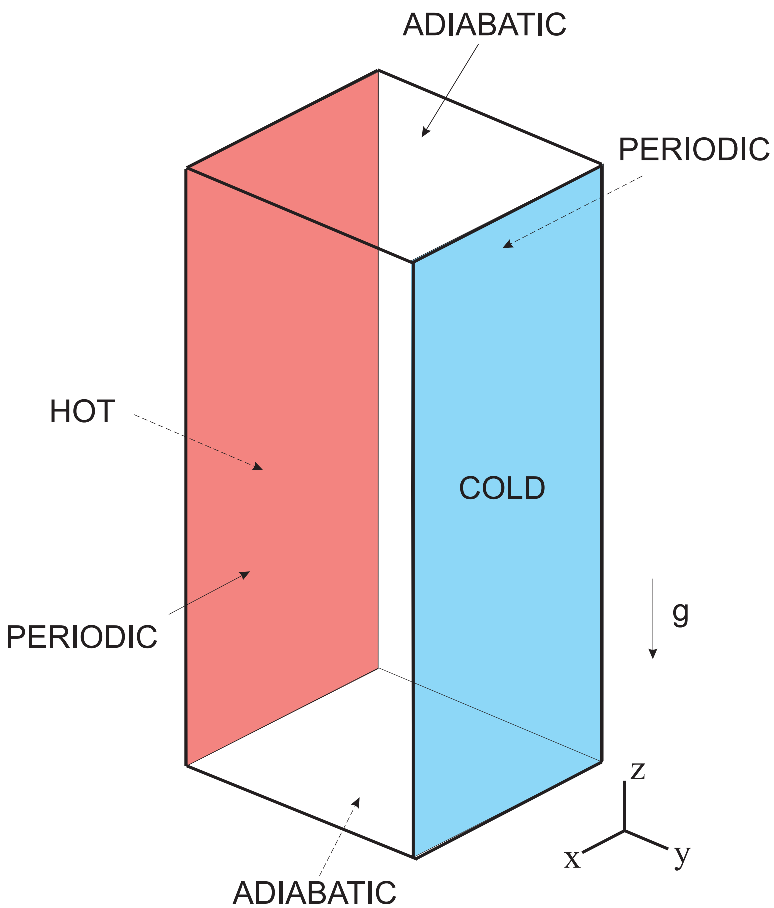
[*] Institute of Mathematics and Computing Science, University of Groningen
P.O. Box 800, 9700 AV Groningen, The Netherlands, E-mail: R.W.C.P.Verstappen@rug.nl

[⋆] Centre Tecnològic de Transferència de Calor (CTTC), Technical University of Catalonia
C/ Colom 11, 08222 Terrassa, Barcelona, Spain, E-mail: cttc@cttc.upc.edu

Presentation outline

1. **Introduction**
   - Problem definition: Differentially Heated Cavity
   - DNS results for $Ra = 10^{11}$, $Pr = 0.71$
   - Governing equations
2. **Regularization models for the simulation of turbulence**
   - Existing regularization: Leray and Navier-Stokes-α models
   - Symmetry-preserving regularization models
   - Mathematical foundation
   - Discretizing the $\mathcal{C}_n$ regularization modelling
3. **Results for a Differentially Heated Cavity**
   - Description of cases
   - Initial test: trial-and-error
   - Parameter-free approach
     - ⋆ Comparison with convergence studies
     - ⋆ Mean fields
4. **Conclusions and Future Research**

Problem definition: Differentially Heated Cavity



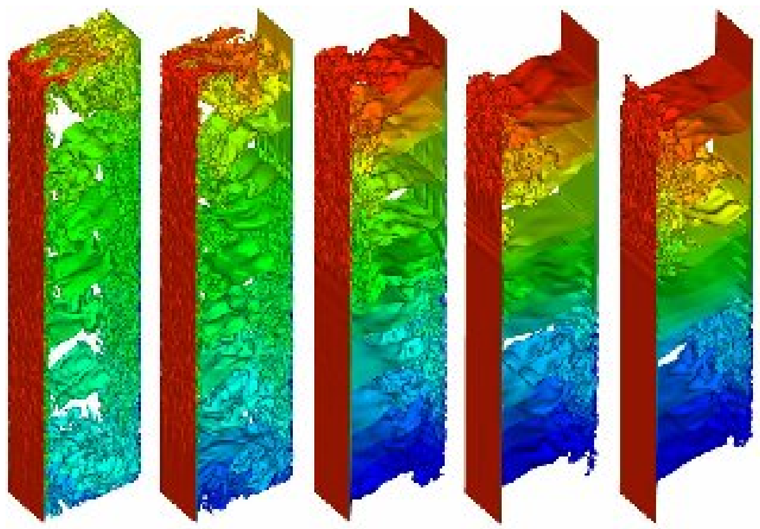
Boundary conditions:

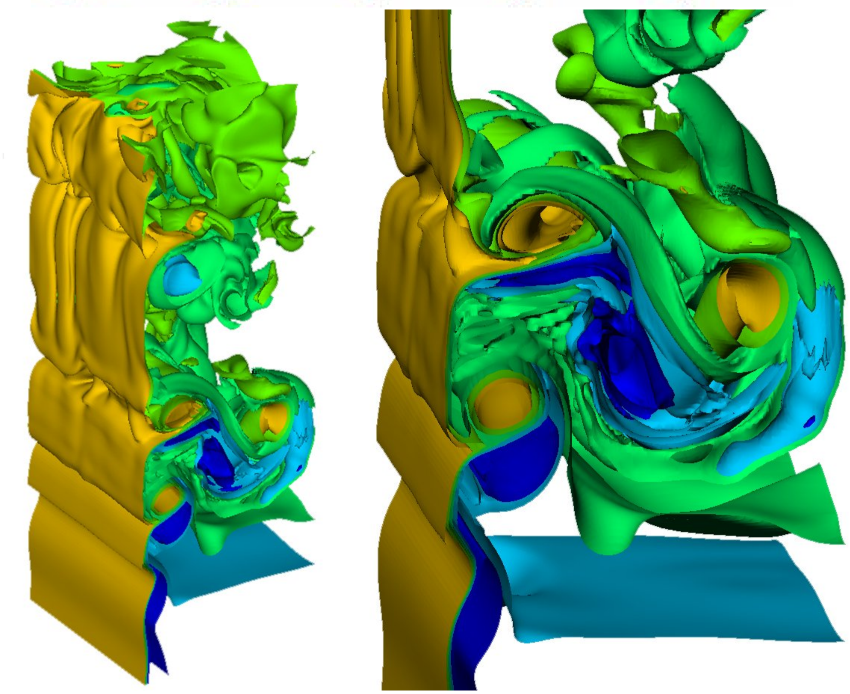
- **Isothermal vertical walls**
- **Adiabatic horizontal walls**
- **Periodic** boundary conditions in the x-direction, orthogonal to the main flow

Dimensionless governing numbers:

- $Ra_z = \frac{\beta \Delta T L_z^3 g}{\alpha \nu}$
- $Pr = \frac{\nu}{\alpha}$
- Height aspect ratio $A_z = \frac{L_z}{L_y}$
- Depth aspect ratio $A_x = \frac{L_x}{L_y}$

DNS results for $Ra = 10^{11}$, $Pr = 0.71$





Some details about **DNS simulations**:

- Mesh size: $128 \times 682 \times 1278$
- Computing Time: ≈ 3 months - 256 CPUs
- 4^{th}-order symmetry-preserving discretization
- $A_z = 4$

Complexity of the flow:

- Boundary layers
- Stratified cavity core
- Internal waves
- Recirculation areas

Centre Tecnològic de Transferència de Calor
UNIVERSITAT POLITÈCNICA DE CATALUNYA

rijksuniversiteit groningen

Governing equations

Incompressible Navier-Stokes coupled with energy transport equation:

$$
\begin{aligned}
\nabla \cdot u &= 0 \\
\partial_t u + \mathcal{C}(u,u) &= Pr\mathcal{D}(u) - \nabla p + f \\
\partial_t T + \mathcal{C}(u,T) &= \mathcal{D}(T)
\end{aligned}
$$

where $f = (0, 0, RaPrT)$ (Boussinesq approximation) and the **nonlinear convective term** is given by

$$\mathcal{C}(u,v) = (u \cdot \nabla)v$$

and the linear dissipative term is given by

$$\mathcal{D}(u) = \frac{1}{Ra^{0.5}}\nabla^2 u$$

Regularization modelling

As the full energy spectrum cannot be computed, a **dynamically less complex mathematical formulation** is sought. We consider smooth approximations (regularizations) of the nonlinearity,

$$\begin{aligned} \partial_t u_\epsilon + \widetilde{\mathcal{C}}(u_\epsilon, u_\epsilon) &= Pr\mathcal{D}(u_\epsilon) - \nabla p_\epsilon + f \\ \partial_t T_\epsilon + \widetilde{\mathcal{C}}(u_\epsilon, T_\epsilon) &= \mathcal{D}(T_\epsilon) \end{aligned}$$

such approximations may fall in the **Large-Eddy Simulation** (LES) concept,

$$\begin{aligned} \partial_t \bar{u}_\epsilon + \mathcal{C}(\bar{u}_\epsilon, \bar{u}_\epsilon) &= Pr\mathcal{D}(\bar{u}_\epsilon) - \nabla \bar{p}_\epsilon + f + \mathcal{M}_1(\bar{u}_\epsilon, \bar{u}_\epsilon) \\ \partial_t \overline{T}_\epsilon + \mathcal{C}(\bar{u}_\epsilon, \overline{T}_\epsilon) &= \mathcal{D}(\overline{T}_\epsilon) + \mathcal{M}_2(\bar{u}_\epsilon, \overline{T}_\epsilon) \end{aligned}$$

if the filter is invertible:

$$\begin{aligned} \mathcal{M}_1(\bar{u}_\epsilon, \bar{u}_\epsilon) &= \mathcal{C}(\bar{u}_\epsilon, \bar{u}_\epsilon) - \overline{\widetilde{\mathcal{C}}(u_\epsilon, u_\epsilon)} \\ \mathcal{M}_2(\bar{u}_\epsilon, \overline{T}_\epsilon) &= \mathcal{C}(\bar{u}_\epsilon, \overline{T}_\epsilon) - \overline{\widetilde{\mathcal{C}}(u_\epsilon, T_\epsilon)} \end{aligned}$$

Previous regularization modellings

Leray and Navier-Stokes-α models

The regularization methods basically **alters the convective term** to **restrain the production of small scales** of motion.

- Leray model:

$$\partial_t u_\epsilon + \mathcal{C}(\bar{u}_\epsilon, u_\epsilon) = \mathcal{D}(u_\epsilon) - \nabla p_\epsilon$$

- Navier-Stokes-α model:

$$\partial_t u_\epsilon + \mathcal{C}_r(u_\epsilon, \bar{u}_\epsilon) = \mathcal{D}(u_\epsilon) - \nabla \pi_\epsilon$$

where the $\pi = p + u^2/2$ and the convective operator in rotational form is defined as

$$\mathcal{C}_r(u, v) = (\nabla \times u) \times v$$

However, in doing so some of the **inviscid invariants** (kinetic energy, enstrophy in 2D and helicity in 3D) are **not conserved**.

Symmetry-preserving regularization models (1/2)

In order to conserve the following inviscid invariants

- Kinetic energy

$$\int_{\Omega} \boldsymbol{u} \cdot \boldsymbol{u} d\Omega$$

- Enstrophy (in 2D)

$$\int_{\Omega} (\nabla \times \boldsymbol{u}) \cdot (\nabla \times \boldsymbol{u}) d\Omega$$

- Helicity (in 3D)

$$\int_{\Omega} (\nabla \times \boldsymbol{u}) \cdot \boldsymbol{u} d\Omega$$

the **approximate convective operator** has to be **skew-symmetric**:

$$\left(\widetilde{\mathcal{C}}(u,v),w\right) = -\left(\widetilde{\mathcal{C}}(u,w),v\right)$$

Centre Tecnològic de Transferència de Calor
UNIVERSITAT POLITÈCNICA DE CATALUNYA

rijksuniversiteit groningen

Symmetry-preserving regularization models (2/2)

This criterion yields the following class of approximations,

$$\partial_t u_\epsilon + \mathcal{C}_n(u_\epsilon, u_\epsilon) = \mathcal{D}(u_\epsilon) - \nabla p_\epsilon$$

in which the convective term in smoothened according to:

$$\begin{aligned}
\mathcal{C}_2(u,v) &= \overline{\mathcal{C}(\bar{u},\bar{v})} \\
\mathcal{C}_4(u,v) &= \mathcal{C}(\bar{u},\bar{v}) + \overline{\mathcal{C}(\bar{u},v')} + \overline{\mathcal{C}(u',\bar{v})} \\
\mathcal{C}_6(u,v) &= \mathcal{C}(\bar{u},\bar{v}) + \mathcal{C}(\bar{u},v') + \mathcal{C}(u',\bar{v}) + \overline{\mathcal{C}(u',v')}
\end{aligned}$$

where $u' = u - \bar{u}$ and $\mathcal{C}_n(u,v) = \mathcal{C}(u,v) + \mathcal{O}(\epsilon^n)$ for **any symmetric filter**.

Mathematical foundation

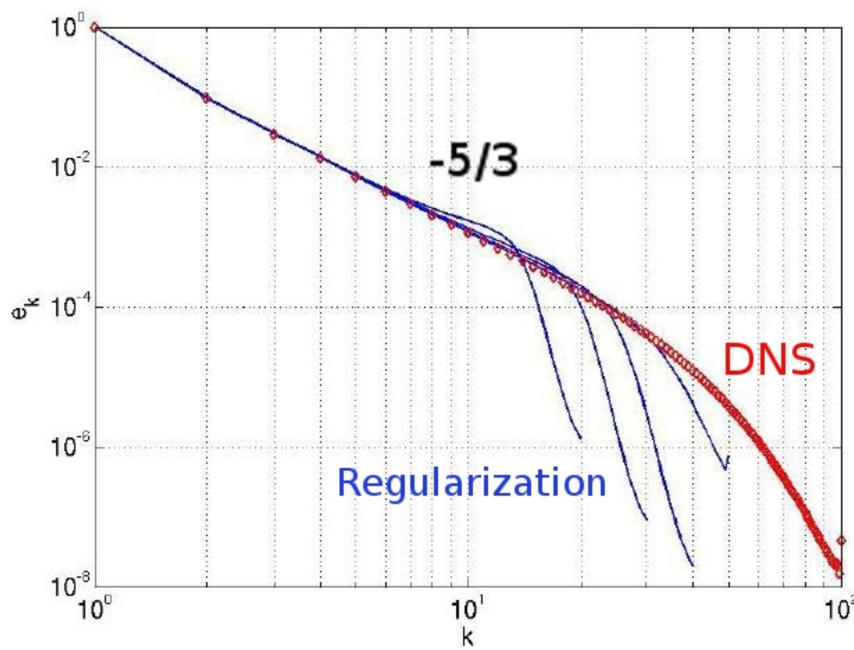
Energy flux equation for the symmetry-preserving regularization resembles the NS

$$\frac{1}{2}\frac{d}{dt}\left|u_{kk'}\right|^2 + \nu\left|\nabla u_{kk'}\right|^2 = \widetilde{T}_k - \widetilde{T}_{k'} \quad \longrightarrow \quad \nu < \left|\nabla u_{kk'}\right|^2 > = < \widetilde{T}_k > - < \widetilde{T}_{k'} >$$

$\Longrightarrow$ Following the same steps as Foias *et al.* (2001)

- $< \widetilde{T}_k >$ is a nonnegative, monotone decreasing function.
- $< \widetilde{T}_k >$ is approximately constant for $k_a < k < k_b$ (existence of inertial range).

$\Longrightarrow$ **$-5/3$ scaling !!!**



Centre Tecnològic de Transferència de Calor
UNIVERSITAT POLITÈCNICA DE CATALUNYA

rijksuniversiteit groningen

LES-interpretation of $\mathcal{C}_4$-regularization

$$\begin{aligned} \partial_t \bar{u}_\epsilon + \mathcal{C}(\bar{u}_\epsilon, \bar{u}_\epsilon) - \mathcal{D}(\bar{u}_\epsilon) + \nabla \bar{p}_\epsilon &= \\ \mathcal{C}(\bar{u}_\epsilon, \bar{u}_\epsilon) - \overline{\mathcal{C}_4(u_\epsilon, u_\epsilon)} &= \\ -\frac{\epsilon^2}{12}\nabla \cdot (\nabla \bar{u}_\epsilon : \nabla \bar{u}_\epsilon) &\quad + \quad \mathcal{O}(\epsilon^4) \\ \text{gradient model} &\quad + \quad \text{stabilization} \end{aligned}$$

Discretizing the $\mathcal{C}_n$ regularization modelling

- The discretization is also a regularization. The **spatial discretization** method preserves the symmetry and conservation properties too

$$\Omega_s \frac{d\boldsymbol{u}_s}{dt} + \mathsf{C}(\boldsymbol{u}_s)\boldsymbol{u}_s + \mathsf{D}\boldsymbol{u}_s + \Omega_s \mathsf{G}\boldsymbol{p}_c = \mathbf{0}_s \qquad \text{with} \quad \mathsf{C}(\boldsymbol{u}_s) = -\mathsf{C}^*(\boldsymbol{u}_s)$$

and is therefore well-suited to test the proposed regularization model.

- A normalized self-adjoint **filter** has been chosen. In 1D it becomes

$$\overline{\phi}_i = \frac{\epsilon^4 - 4\epsilon^2}{1152}(\phi_{i+2} + \phi_{i-2}) + \frac{16\epsilon^2 - \epsilon^4}{288}(\phi_{i+1} + \phi_{i-1}) + \frac{\epsilon^4 - 20\epsilon^2 + 192}{192}\phi_i$$

Results for differentially heated cavity at $Ra = 10^{11}$

- Regularization model $\mathcal{C}_4$ is tested.
- Two coarse meshes are considered

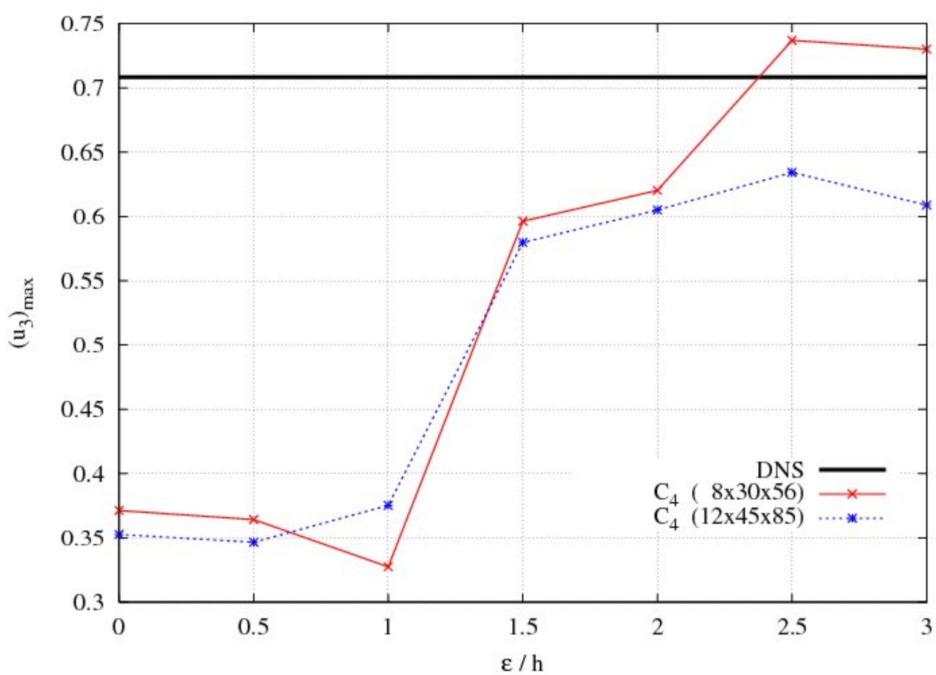
	DNS	RM1	RM2
Nx	128	12	8
Ny	682	45	30
Nz	1278	85	56
Δx_{min}	3.79×10^{-3}	4.16×10^{-2}	6.25×10^{-2}
Δy_{min}	2.16×10^{-4}	3.27×10^{-3}	4.91×10^{-3}
Δz_{min}	3.13×10^{-3}	4.71×10^{-2}	7.14×10^{-2}

- **Initial test:** ratio ϵ/h (filter length to the average grid width) is kept constant in all three spatial directions.

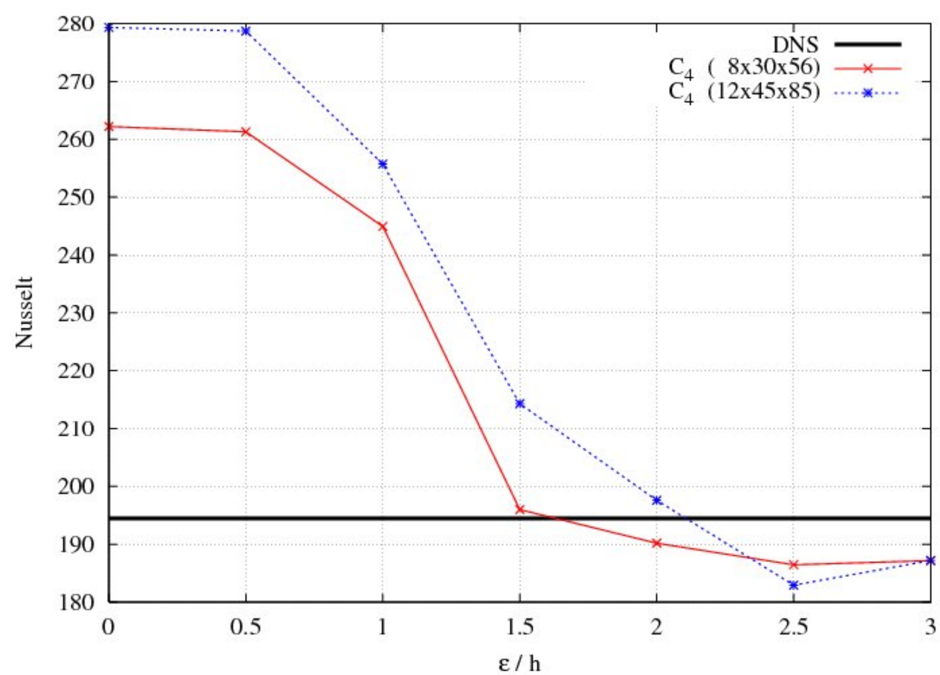
Results for differentially heated cavity at $Ra = 10^{11}$

Convergence studies

w_{max}



Nusselt



The maximum of the averaged vertical velocity at the horizontal mid-height plane and the overall averaged Nusselt number as a function of the ratio of the filter length ϵ to the average grid width h.

$\Longrightarrow$ A **weak dependance** for sufficiently **large values of** ϵ is observed

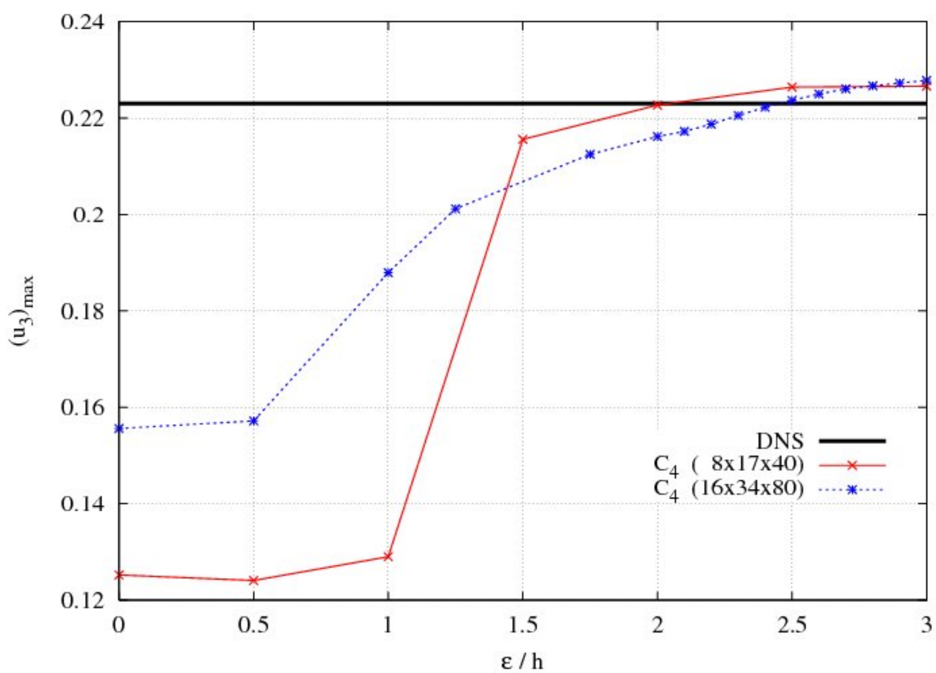
Centre Tecnològic de Transferència de Calor
UNIVERSITAT POLITÈCNICA DE CATALUNYA

rijksuniversiteit groningen

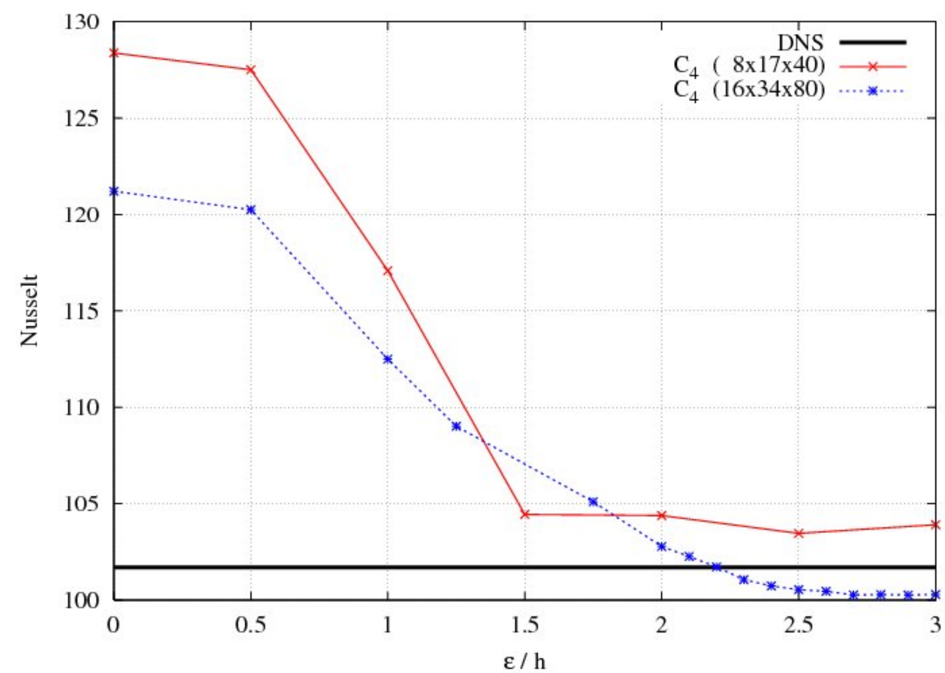
Same behaviour has also been observed at different Ra-numbers!

Convergence studies at $Ra = 10^{10}$

w_{max}



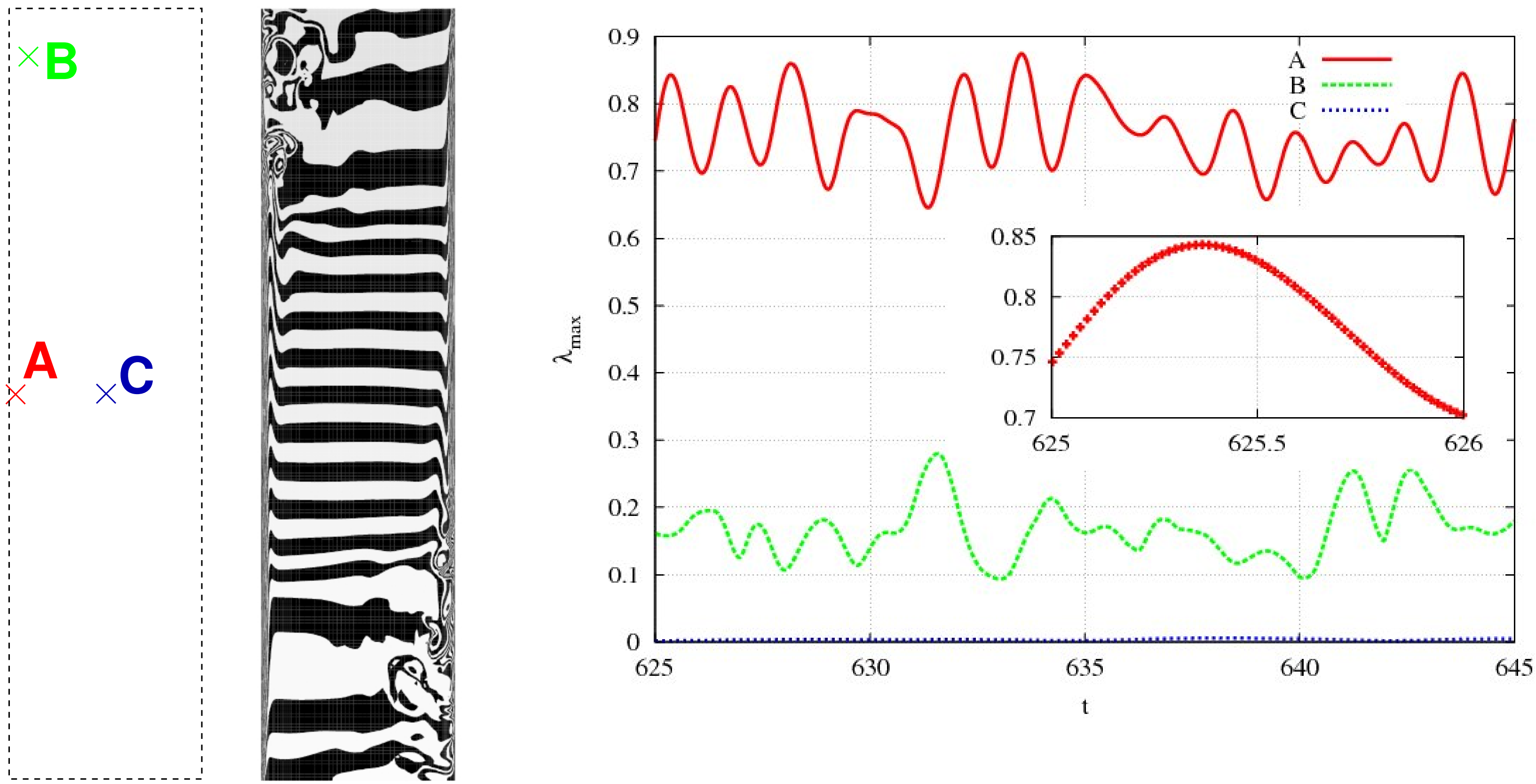
Nusselt



The maximum of the averaged vertical velocity at the horizontal mid-height plane and the overall averaged Nusselt number as a function of the ratio of the filter length ϵ to the average grid width h.

$\Longrightarrow$ A **weak dependance** for sufficiently **large values of** ϵ is observed

Parameter-free approach



Centre Tecnològic de Transferència de Calor
UNIVERSITAT POLITÈCNICA DE CATALUNYA

rijksuniversiteit groningen

Parameter-free approach

The **vortex-stretching** and **dissipation term** contributions to $(1/|\omega|^2)\partial_t|\omega|^2$ are given by

$$\frac{\omega \cdot \mathcal{C}\left(\omega, u\right)}{\omega \cdot \omega} = \frac{\omega \cdot \mathcal{S}\left(u\right)\omega}{\omega \cdot \omega} \quad \text{and} \quad \frac{1}{Re}\frac{\nabla\omega : \nabla\omega}{\omega \cdot \omega}$$

At the **smallest grid scale**, $k = \pi/h$, **convection** may **dominate** diffusion

$$\frac{\omega_k \cdot \mathcal{C}\left(\omega, u\right)_k}{\omega_k \cdot \omega_k} \quad > \quad \frac{1}{Re}k^2$$

$\Longrightarrow$ In the present work we **determine the filter width** ϵ from

$$\frac{\omega_k \cdot \mathcal{C}_4\left(\omega, u\right)_k}{\omega_k \cdot \omega_k} \quad \approx \quad \frac{1}{Re}k^2$$

Centre Tecnològic de Transferència de Calor
UNIVERSITAT POLITÈCNICA DE CATALUNYA

rijksuniversiteit groningen

Parameter-free approach

Note that $\mathcal{C}_4(u,v)$ depends on the filter length ϵ. For the **smallest scale** this dependence becomes

$$\frac{\omega_k \cdot \mathcal{C}_4(\omega,u)_k}{\omega_k \cdot \omega_k} \approx f_4(\hat{g}_k(\epsilon)) \frac{\omega_k \cdot \mathcal{S}(u)\,\omega_k}{\omega_k \cdot \omega_k} \leq f_4(\hat{g}_k(\epsilon))\,\lambda_{max}(\mathcal{S})$$

where $0 < \hat{g}_k(\epsilon) \leq 1$ is the transfer function of the filter and the damping function $0 < f_4 \leq 1$.

$\Longrightarrow$ Therefore, it suffices that following inequality be **locally** hold

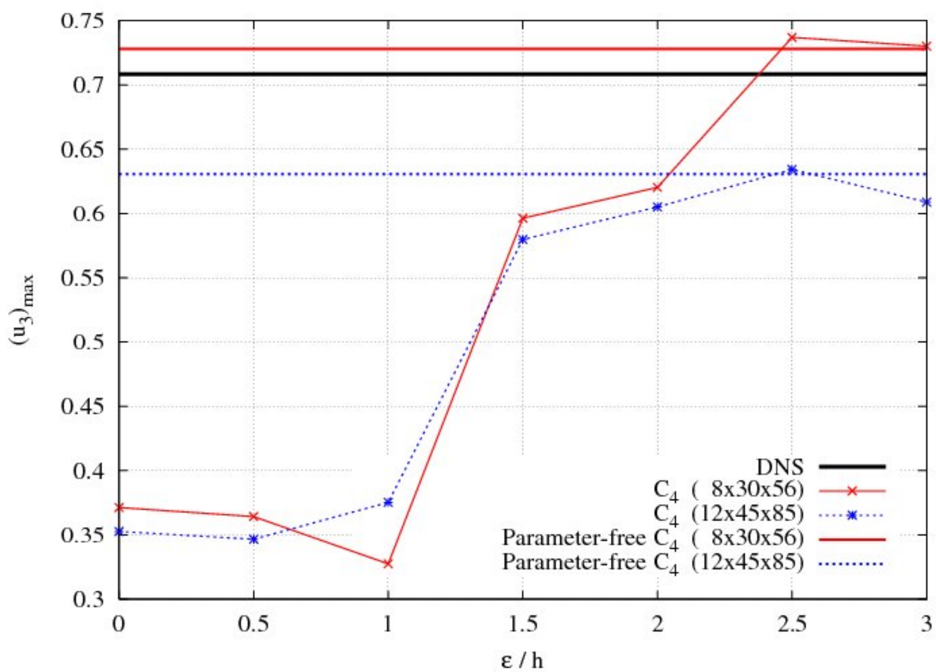
$$f_4(\hat{g}_k(\epsilon)) \leq \frac{1}{Re}\frac{k^2}{\lambda_{max}(\mathcal{S})} \qquad \longrightarrow \qquad \epsilon$$

to guarantee that the **production of smaller scales of motion be stopped at the smallest scale** set by the mesh.

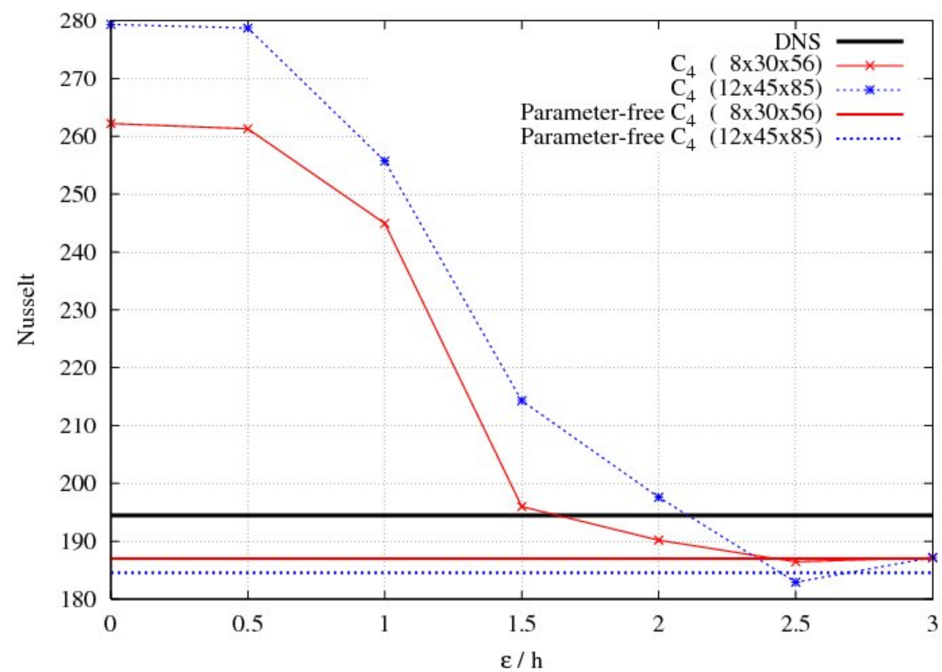
Results for differentially heated cavity at $Ra = 10^{11}$

Free-parameter approach

w_{max}



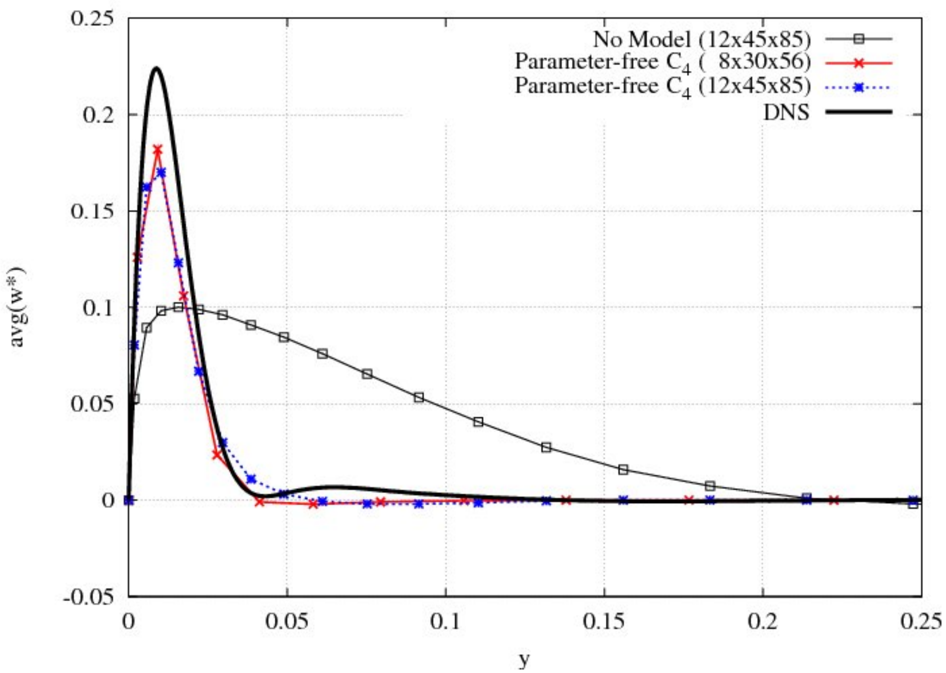
Nusselt

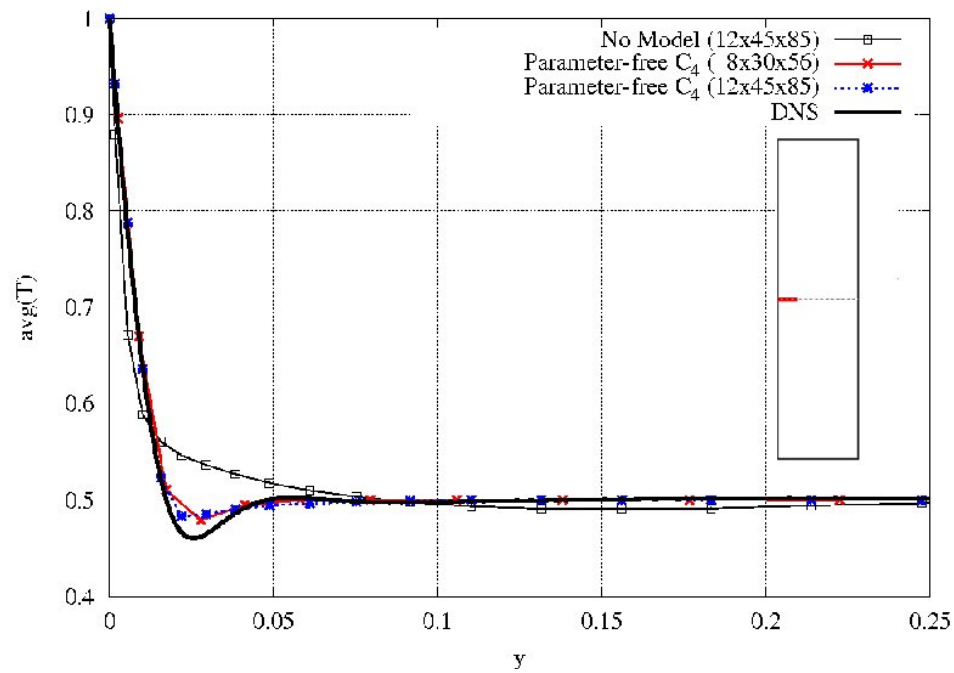


The maximum of the averaged vertical velocity at the horizontal mid-height plane and the overall averaged Nusselt number as a function of the ratio of the filter length ϵ to the average grid width h.

Results for differentially heated cavity at $Ra = 10^{11}$

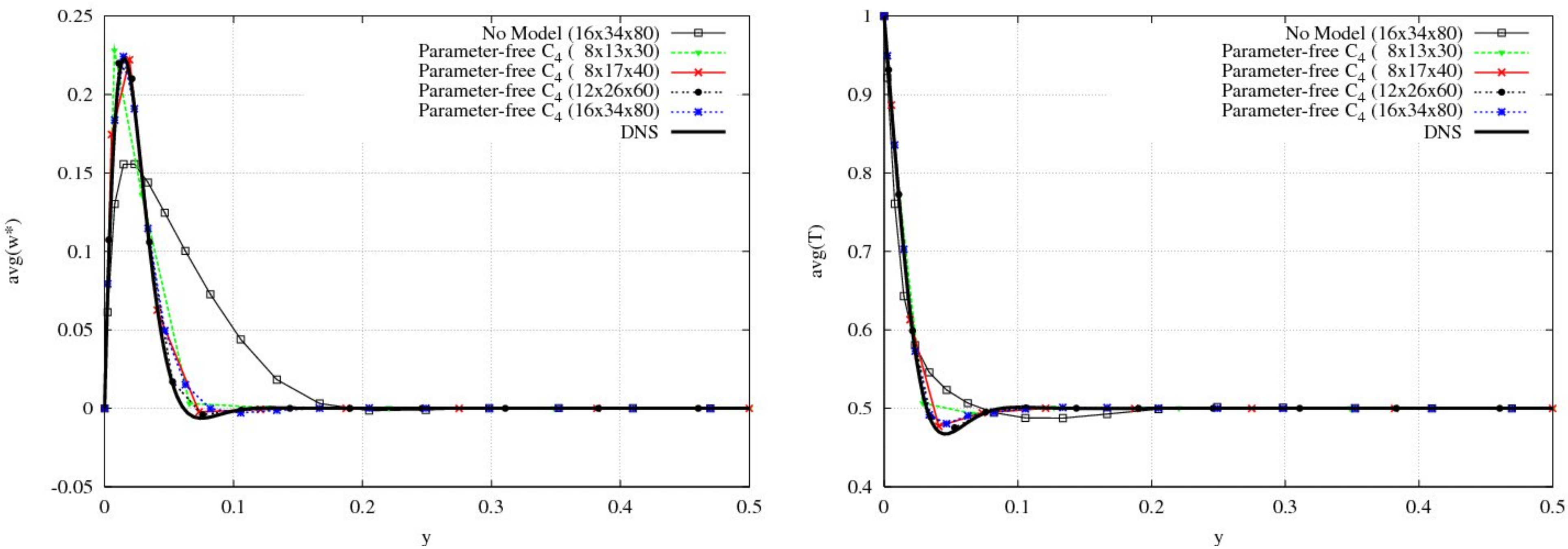
Profiles





Averaged vertical velocity and temperature profiles at the horizontal mid-height plane.

How does the parameter-free $\widetilde{\mathcal{C}}_4$ symmetry-preserving regularization modelling behave for other grids and Ra-numbers?

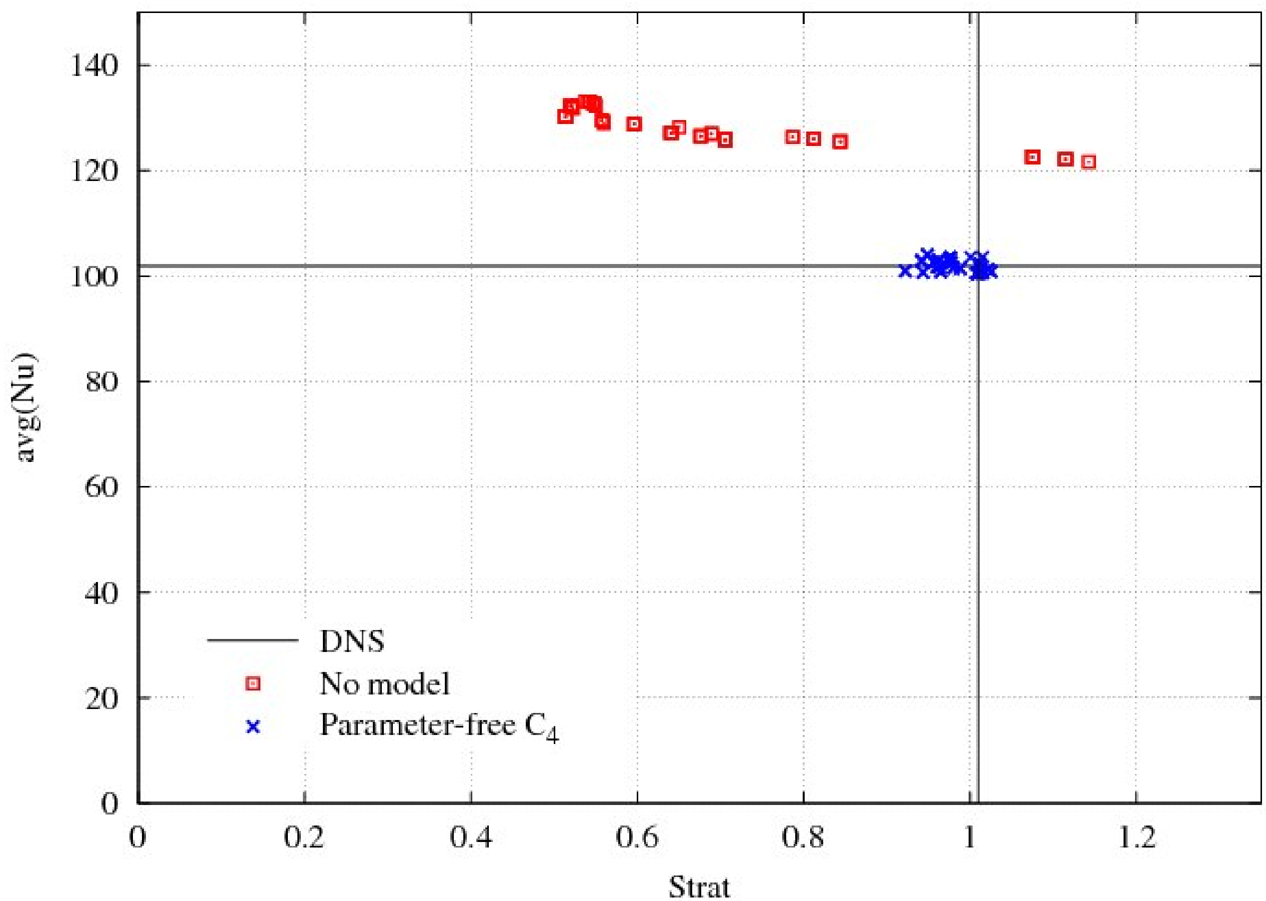


Averaged vertical velocity and temperature profiles at the horizontal mid-height plane at $Ra = 10^{10}$.

Even for a **very coarse** $8 \times 13 \times 30$ grid **reasonable results** are being obtained!

$\Longrightarrow$ Results for different grids show the **robustness** of the method.

A challenging test: mesh independence analysis at very coarse grids



The overall Nusselt number and the centerline stratification for 50 randomly generated coarse grids with fixed stretching at $Ra = 10^{10}$. $8 \leq N_x \leq 16$, $17 \leq N_y \leq 34$, and $40 \leq N_z \leq 80$.

Conclusions and Future Research

The results shown illustrate the potential of the conservative parameter-free $\widetilde{\mathcal{C}}_4$ smoothing as a new simulation shortcut.

The main advantages with respect exiting LES models can be summarized:

- **Robustnest**. As the smoothed governing equations preserve the symmetry properties of the original Navier-Stokes equations the solution cannot blow up (in the energy-norm, in 2D also: enstrophy-norm). It seems that even for very coarse meshes reasonably results can be obtained.
- **Universality**. No *ad hoc* phenomenological arguments that can not be formally derived for the Navier-Stokes equations are used.
- The proposed method constitutes a **parameter-free turbulence model**.

Since now, the method has been **successfully tested** on completely different turbulent configurations such as:

- Channel flow.
- Flow around a wall-mounted cube.
- **Differentially heated cavity** at different Ra-numbers.

Thank you for you attention

Discretization of the convective operator: a symmetry-preserving discretization

The spatially discrete incompressible Navier-Stokes equations are expressed as

$$\Omega_s \frac{d\boldsymbol{u}_s}{dt} + \mathsf{C}\left(\boldsymbol{u}_s\right)\boldsymbol{u}_s + \mathsf{D}\boldsymbol{u}_s + \Omega_s \mathsf{G}\boldsymbol{p}_c = \mathbf{0}_s$$

$$\mathsf{M}\boldsymbol{u}_s = \mathbf{0}_c$$

$\Longrightarrow$ It was shown that the **convective matrix** $\mathsf{C}\left(\boldsymbol{u}_s\right)$ has to be **skew-symmetric**,

$$\mathsf{C}\left(\boldsymbol{u}_s\right) + \mathsf{C}^*\left(\boldsymbol{u}_s\right) = 0$$

... to **preserve** the continuous **invariants** (kinetic energy, enstrophy in 2D and helicity in 3D) in a **discrete sense**.

Choice of the filter

Let us consider a generic linear filter

$$\bar{u}_\epsilon = F u_\epsilon$$

Then, three basic properties are required for the filter:

$$\begin{aligned} \bar{u}_\epsilon &= u_\epsilon + \mathcal{O}(\epsilon^2) \\ (\Omega_s F) &= (\Omega_s F)^* \\ F1 &= 1 \end{aligned}$$

$\Longrightarrow$ Our discrete filter is a 5-**point Gaussian filter.**