Building proper invariants for subgrid-scale eddy-viscosity models

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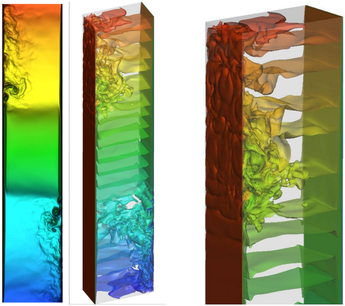
Contents

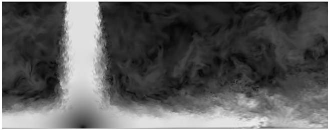
DNS of turbulent incompressible flows

Main features of the DNS code:

- Structured staggered grids
- High-order symmetry-preserving schemes
- Fully-explicit second-order time-integration method
- Poisson solver for 2.5D problems: FFT + PCG
- Hybrid MPI+OpenMP parallelization
- OpenCL-based extension for its use on GPGPU



Air-filled differentially heated cavity at $Ra = 10^{11}$ (111M grid points), **2008**



Plane impingement jet at $Re = 20000$ (102M grid points), **2011**

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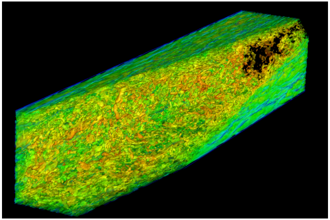
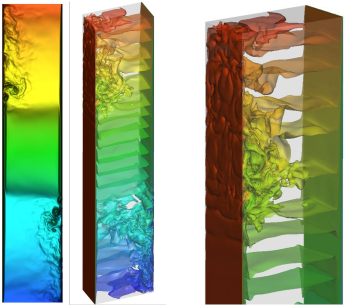


Square duct at $Re_\tau = 1200$ (172M grid points), **2013**



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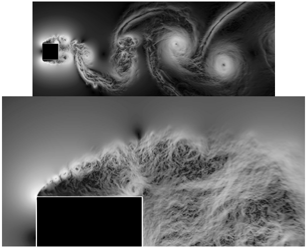
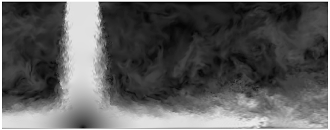


Square cylinder at $Re = 22000$ (324M grid points), **2014**



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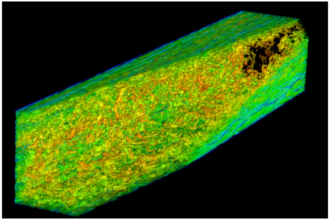

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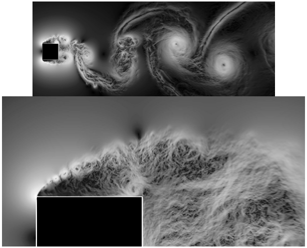

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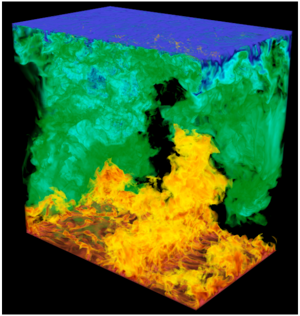
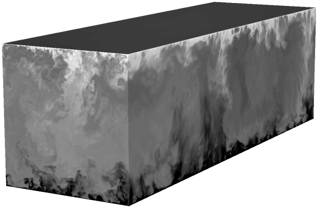
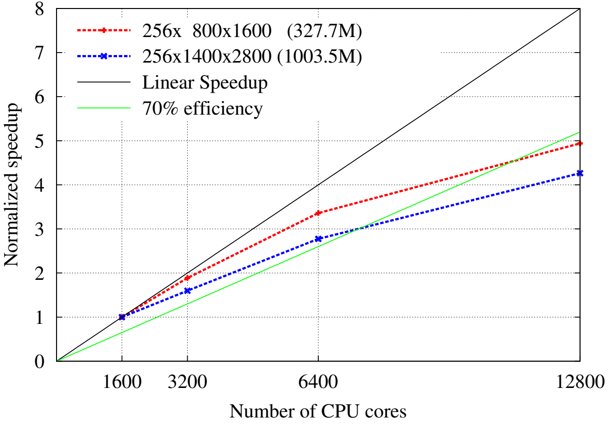




Rayleigh-Bénard convection at $Ra = 10^{10}$ (607M grid points), **2015**

Scaling is possible[1]... but never enough



[1] A.Gorobets, F.X.Trias, A.Oliva. *A parallel MPI+OpenMP+OpenCL algorithm for hybrid supercomputations of incompressible flows*, **Computers&Fluids**, 88:764-772, 2013

Building proper invariants for LES models

Many turbulence **eddy-viscosity models** for LES have been proposed

$$\partial_t \overline{u} + \mathcal{C}(\overline{u}, \overline{u}) = \mathcal{D}\overline{u} - \nabla \overline{p} - \nabla \cdot \tau(\overline{u}) \quad ; \qquad \nabla \cdot \overline{u} = 0$$

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Therefore, they can be characterized by 5 **basic invariants**

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Notation: given a second-order tensor A

First invariant:	$P_A = tr(A)$
Second invariant:	$Q_A = 1/2\{tr^2(A) - tr(A^2)\}$
Third invariant:	$R_A = det(A)$

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Notation:

$$V^2 = 4(tr(S^2\Omega^2) - 2Q_S Q_\Omega),$$

where $S = 1/2(G + G^T)$ and $\Omega = 1/2(G - G^T)$.

A unified framework for eddy-viscosity models

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Smagorinsky model $\quad \nu_e^{Smag} = (C_S\Delta)^2|S(\overline{u})| = 2(C_S\delta)^2(-Q_S)^{1/2},$

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Sigma model $\quad \nu_e^{\sigma} = (C_\sigma\Delta)^2\dfrac{\sigma_3(\sigma_1-\sigma_2)(\sigma_2-\sigma_3)}{\sigma_1^2},$

where $\sigma_i = \sqrt{\lambda_i}$ and λ_i is an eigenvalue of GG^T.

Near-wall behavior

$$\boxed{\{Q_S, R_S, Q_G, R_G, V^2\}}$$

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	Invariants					
	Q_G	R_G	Q_S	R_S	V^2	Q_Ω
Wall-behavior	$\mathcal{O}(y^2)$	$\mathcal{O}(y^3)$	$\mathcal{O}(y^0)$	$\mathcal{O}(y^1)$	$\mathcal{O}(y^0)$	$\mathcal{O}(y^0)$
Units	$[T^{-2}]$	$[T^{-3}]$	$[T^{-2}]$	$[T^{-3}]$	$[T^{-4}]$	$[T^{-2}]$

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Hence, new models can be derived by imposing restrictions on the differential operators they are based on.

Building proper invariants for LES models[2]

For instance, let us consider models that are based on the invariants of the tensor GG^T

$$\nu_e = (C_M \delta)^2 P^p_{GG^T} Q^q_{GG^T} R^r_{GG^T},$$

[2] F.X.Trias, D.Folch, A.Gorobets, A.Oliva. *Building proper invariants for eddy-viscosity subgrid-scale models*, **Physics of Fluids**, 27: 065103, 2015.

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$$-6r - 4q - 2p = -1; \qquad 6r + 2q = s,$$

where s is the slope for the asymptotic near-wall behavior, *i.e.* $\mathcal{O}(y^s)$.

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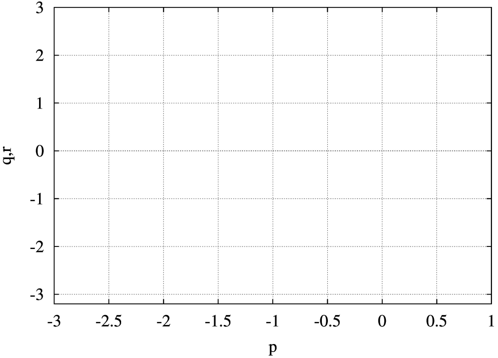
Building proper invariants for LES models

Solutions: $q(p,s) = (1-s)/2 - p$ and $r(p,s) = (2s-1)/6 + p/3$



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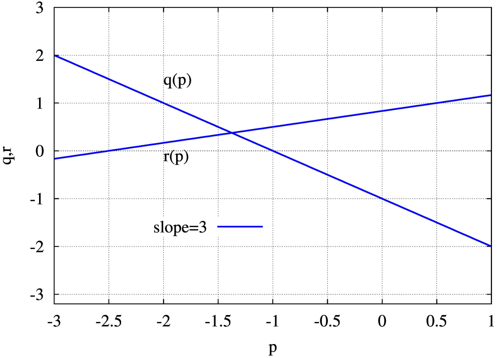

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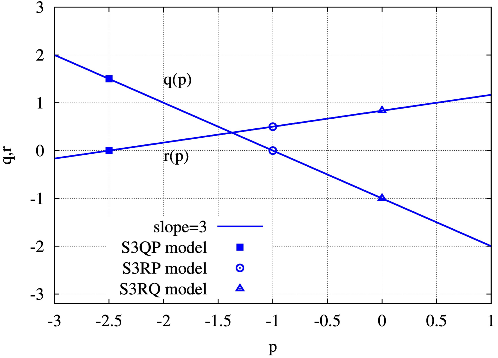

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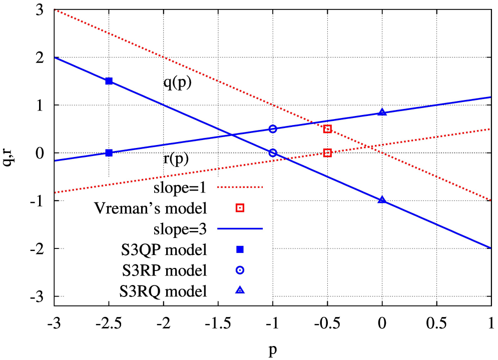

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has been derived by **imposing proper conditions** on the invariant(s)

$$\begin{aligned}
\nu_e^{S3QP} &= (C_{s3qp}\delta)^2 P_{GG^T}^{-5/2} Q_{GG^T}^{3/2}, \\
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And what about the model constants?

Building proper models for LES

Finding model constants

The model constants, C_{s3xx}, can be related with the Vreman's constant, C_{Vr}, with the following inequality

$$0 \leq \frac{(C_{Vr})^2}{(C_{s3xx})^2} \frac{\nu_e^{S3xx}}{\nu_e^{Vr}} \leq \frac{1}{3}.$$

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numerical stability

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$$0 \leq \nu_e^{s3xx} \leq \nu_e^{Vr}$$

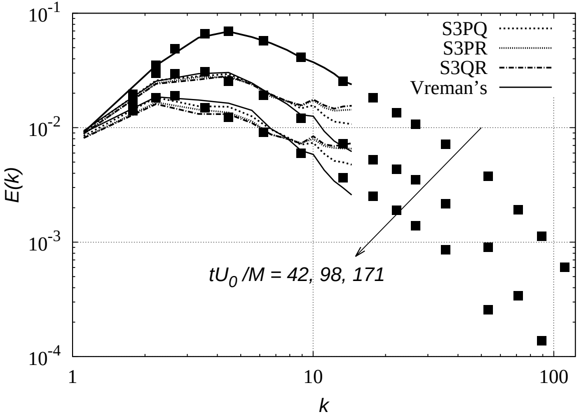
numerical stability

less or equal dissipation than Vreman's model!

Buiding proper models for LES

Decaying isotropic turbulence with $C_{s3pq} = C_{s3pr} = C_{s3qr} = \sqrt{3}C_{Vr}$

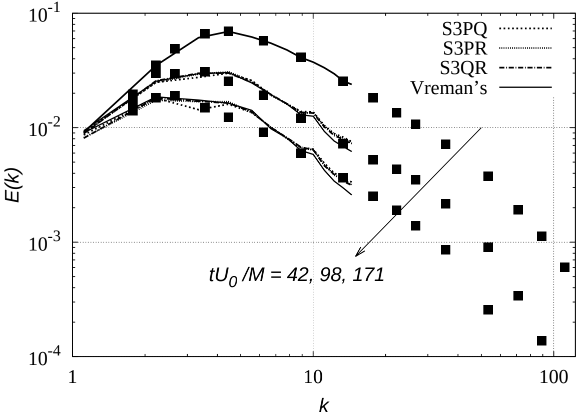
Comparison with classical Comte-Bellot & Corrsin (CBC) experiment.



Buiding proper models for LES

Decaying isotropic turbulence with $C_{s3pq} = 0.572,\ C_{s3pr} = 0.709,\ C_{s3qr} = 0.762$

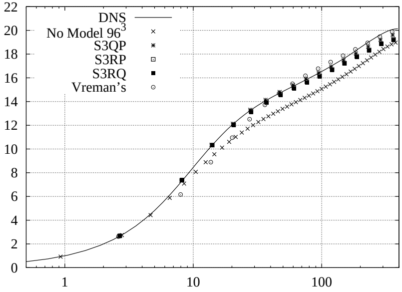
Comparison with classical Comte-Bellot & Corrsin (CBC) experiment.



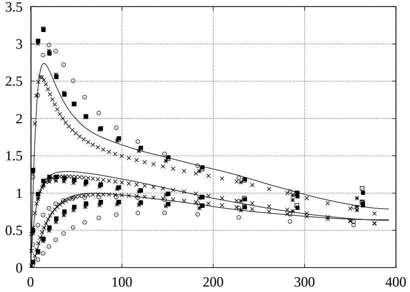
Turbulent channel flow

Results

$Re_\tau = 395$ DNS Moser et al. LES 32^3



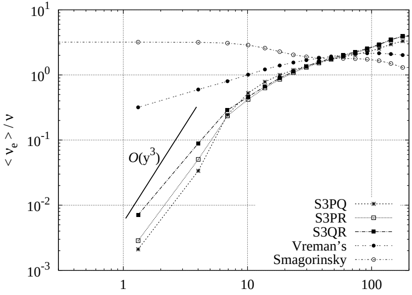
mean velocity



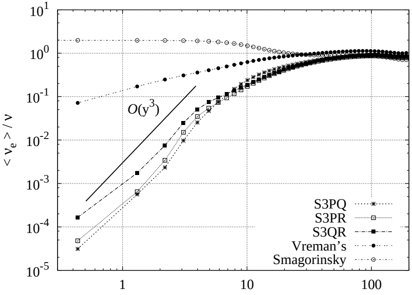
rms fluctuations

Turbulent channel flow

Near-wall behavior



32x32x32



32x96x32

Conclusions

- Most of the existing eddy-viscosity models for LES can be represented into this 5D phase space of invariants

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- Most of the existing eddy-viscosity models for LES can be represented into this 5D phase space of invariants

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- Based on this general framework, a family of **new eddy-viscosity type** LES models has been derived by imposing proper restrictions.

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Conclusions and Future Research

- Most of the existing eddy-viscosity models for LES can be represented into this 5D phase space of invariants

$$\boxed{\{Q_S, R_S, Q_G, R_G, V^2\}}$$

- Based on this general framework, a family of **new eddy-viscosity type** LES models has been derived by imposing proper restrictions.

- Test the performance of new eddy-viscosity type LES for different configurations.

Thank you for your attention

Further reading

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Building proper invariants for eddy-viscosity subgrid-scale models

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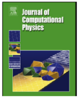

A simple approach to discretize the viscous term with spatially varying (eddy-)viscosity

CrossMark

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