Symmetry-preserving regularization of wall-bounded turbulent flows

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Governing equations

Incompressible Navier-Stokes equations:

\[ \nabla \cdot u = 0 \]
\[ \partial_t u + C(u, u) = D(u) - \nabla p \]

where the **nonlinear convective term** is given by

\[ C(u, \phi) = (u \cdot \nabla)\phi \]

and the linear dissipative term is given by

\[ D(\phi) = \nu \Delta \phi \]
Regularization modeling

As the full energy spectrum cannot be computed, a dynamically less complex mathematical formulation is sought. We consider smooth approximations (regularizations) of the nonlinearity,

$$\partial_t u_\epsilon + \tilde{C}(u_\epsilon, u_\epsilon) = \mathcal{D}(u_\epsilon) - \nabla p_\epsilon$$

such approximations may fall in the Large-Eddy Simulation (LES) concept,

$$\partial_t \bar{u}_\epsilon + C(\bar{u}_\epsilon, \bar{u}_\epsilon) = \mathcal{D}(\bar{u}_\epsilon) - \nabla \bar{p}_\epsilon + \mathcal{M}_1(\bar{u}_\epsilon, \bar{u}_\epsilon)$$

if the filter is invertible:

$$\mathcal{M}_1(\bar{u}_\epsilon, \bar{u}_\epsilon) = C(\bar{u}_\epsilon, \bar{u}_\epsilon) - \tilde{C}(u_\epsilon, u_\epsilon)$$
Symmetry-preserving regularization models

In order to conserve the following inviscid invariants

- Kinetic energy: \((u, u)\)
- Enstrophy (in 2D): \((\omega, \omega)\)
- Helicity (in 3D): \((\omega, u)\)

where \((a, b) = \int_{\Omega} a \cdot b \, d\Omega\) and \(\omega = \nabla \times u\); the approximate convective operator must preserve the basic symmetry properties:

\[
(C(u, v), w) = - (C(u, w), v) \\
(C(u, v), \Delta v) = (C(u, \Delta v), v) \quad \text{in 2D}
\]
Symmetry-preserving regularization models

Regularizations of the non-linear convective term can be constructed

$$\tilde{\mathcal{C}}(u, v) = \sum_{i,j,k=0}^{1} a_{ijk} \tilde{\mathcal{C}}_{ijk}(u, v)$$

where $\tilde{\mathcal{C}}_{ijk}(u, v) = \varphi_k (C(\varphi_i(u), \varphi_j(v)))$ and $\varphi_i(u) = \begin{cases} u & \text{if } i = 0 \\ \frac{u}{u} & \text{if } i = 1 \end{cases}$

$\overline{(\cdot)}$ is a **self-adjoint** filter that **commutes** with differential operators.

Among all possible combinations we find the regularization proposed by Leray, $C(\bar{u}, u) : a_{100} = 1$ (with the rest of $a_{ijk} = 0$)

$\implies$ Eight coefficients $a_{ijk}$ need to be determined.
Symmetry-preserving regularization models

\[
\tilde{C}(u, v) = \sum_{i,j,k=0}^{1} a_{ijk} \tilde{C}_{ijk}(u, v)
\]

\[
\sum_{i,j,k=0}^{1} a_{ijk} = 1 \quad \rightarrow \quad \tilde{C}(u, v) = C(u, v) + \mathcal{O}(\epsilon^n) \quad \text{with } n \geq 2
\]

\[
(\tilde{C}(u, v), w) = - (\tilde{C}(u, w), v) \quad \rightarrow \quad a_{ijk} = a_{ikj}
\]

\[
(\tilde{C}(u, v), \Delta v) = (\tilde{C}(u, \Delta v), v) \quad \text{in 2D} \quad \rightarrow \quad a_{ijk} = a_{kji}
\]

This leads to a family of \(\mathcal{O}(\epsilon^2)\)-accurate regularizations. Among them\(^1\),

\[
C_2(u, v) = \tilde{C}_{111}(u, v) = \overline{C(u, v)}
\]

Symmetry-preserving regularization models

To cancel second-order terms, three additional conditions need to imposed:

\[
\sum_{j,k=0}^1 a_{1jk} = 0 \quad \sum_{i,k=0}^1 a_{i1k} = 0 \quad \sum_{i,j=0}^1 a_{ij1} = 0
\]

\[
C_4^\gamma(u, v) = \frac{1}{2} ((C_4 + C_6) + \gamma(C_4 - C_6)) (u, v)
\]

where \(C_4\) and \(C_6\) read

\[
C_4(u, v) = C(\bar{u}, \bar{v}) + \overline{C(\bar{u}, v')} + \overline{C(u', \bar{v})}
\]

\[
C_6(u, v) = C(\bar{u}, \bar{v}) + C(\bar{u}, v') + C(u', \bar{v}) + \overline{C(u', v')}
\]
Symmetry-preserving regularization models

Taking $\gamma = 1$ we obtain the $C_4$ approximation\textsuperscript{1},

$$\partial_t u_\epsilon + C_4(u_\epsilon, u_\epsilon) = D(u_\epsilon) - \nabla p_\epsilon$$

in which the convective term in smoothed according to:

$$C_4(u, v) = C(\bar{u}, \bar{v}) + \overline{C(\bar{u}, v')} + \overline{C(u', \bar{v})}$$

where $u' = u - \bar{u}$ and $C_4(u, v) = C(u, v) + O(\epsilon^4)$ for any symmetric filter.

High-frequencies need to be effectively damped.

But how much?

\textsuperscript{1}Roel Verstappen, Computers & Fluids, 37 (7): 887-897, 2008
Stopping the vortex-stretching\textsuperscript{2}

Taking the curl of momentum equation the vorticity transport equation follows

$$\partial_t \omega + \mathcal{C}(u, \omega) = \mathcal{C}(\omega, u) + \mathcal{D}(\omega)$$

Let us now consider an arbitrary part of the flow domain, $\Omega$, with periodic boundary conditions. Then, taking the $L^2$ innerproduct with $\omega = \nabla \times u$ leads to the enstrophy equation

$$\frac{1}{2} \frac{d}{dt} (\omega, \omega) = (\omega, \mathcal{C}(\omega, u)) - \nu (\nabla \omega, \nabla \omega)$$

where $(a, b) = \int_{\Omega} a \cdot b d\Omega$. Unless, the grid is fine enough convection dominates diffusion

$$(\omega, \mathcal{C}(\omega, u)) > \nu (\nabla \omega, \nabla \omega)$$

\textsuperscript{2}F.X. Trias et al. \textit{Computers\&Fluids}, 39:1815-1831, 2010
The vortex-stretching term can be expressed in terms of the invariant
\[ r = -\frac{1}{3} \text{tr}(S^3) \]

whereas the \( L^2(\Omega) \)-norm of \( \omega \) in terms of the invariant \( q = -\frac{1}{2} \text{tr}(S^2) \)

\[ (\omega, \omega) = -4 \int_\Omega q \, d\Omega \]

Then, the diffusive term can be bounded by

\[ \nu (\nabla \omega, \nabla \omega) = -\nu (\omega, \Delta \omega) \leq -\nu \lambda_\Delta (\omega, \omega) = 4\nu \lambda_\Delta \int_\Omega q \, d\Omega \]

where \( \lambda_\Delta < 0 \) is the largest (smallest in absolute value) non-zero eigenvalue of Laplacian operator \( \Delta \) on \( \Omega \). If we now consider that the domain is a periodic box of volume \( h \), then \( \lambda_\Delta = -(\pi/h)^2 \).
In the present work we determine the filter width $\epsilon$ from

$$ (\omega, C_4(\omega, u)) \approx f_4(\hat{g}_k(\epsilon))(\omega, C(\omega, u)) \leq \nu (\nabla \omega, \nabla \omega) $$

Then, recalling identity (1) and inequality (2), we propose to rewrite the previous inequality in terms of the invariants $q$ and $r$

$$ f_4(\hat{g}_k) = \min \left\{ \nu \lambda_\Delta \frac{q}{r^+}, 1 \right\} \quad with \quad r^+ = \max(r, 0) $$

Notice that $q < 0$ (dissipation) whereas $r$ can be either positive or negative.

- Switches off ($f_4 = 1$) for: laminar ($r \to 0$), 2D flows ($r = 0$) and for fine enough meshes, $|\nu \lambda_\Delta q/r| \geq 1$.
- Consistent near-wall behavior $r \propto y^3$ and $q \propto y^0$.
- Consistent with the preferential vorticity alignment with the intermediate eigenvector, $\lambda_2$ (experimentally observed)
Test-case: Differentially Heated Cavity

Boundary conditions:

- Isothermal vertical walls
- Adiabatic horizontal walls
- Periodic boundary conditions in the spanwise direction

Dimensionless governing numbers:

- \( Ra = \beta \Delta T L_z^3 g / (\alpha \nu) \)
- \( Pr = \nu / \alpha \)
- Height aspect ratio \( A_z = L_z / L_y \)
- Depth aspect ratio \( A_x = L_x / L_y \)
DNS\textsuperscript{3,4} results for $Ra = 10^{11}$, $Pr = 0.71$

Some details about DNS:

- Mesh size: $128 \times 682 \times 1278$
- $\approx 3$ months - 256 CPUs
- $4^{th}$-order symmetry-preserving scheme
- $A_z = 4$

Complexity of the flow:

- Boundary layers
- Stratified cavity core
- Internal waves
- Recirculation areas

Results for differentially heated cavity at $Ra = 10^{11}$

- Regularization model $C_4$ is tested.
- Two coarse meshes are considered

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<tr>
<td>$N_z$</td>
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</tr>
</tbody>
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- The **discrete linear filter**\(^5\) is based on polynomial functions of the discrete diffusive operator, $D$

Results for differentially heated cavity at $Ra = 10^{11}$

Profiles

Averaged vertical velocity and temperature profiles at the horizontal mid-height plane.
How does the parameter-free $\tilde{C}_4$ regularization modeling behave for other grids and $Ra$-numbers?

Averaged vertical velocity and temperature profiles at the horizontal mid-height plane at $Ra = 10^{10}$.

Even for a very coarse $8 \times 13 \times 30$ grid reasonable results are obtained!

$\implies$ Results for different grids show the robustness of the method.
Challenging $C_4$: mesh independence analysis

The overall Nusselt number and the centerline stratification for 50 randomly generated coarse grids with fixed stretching at $Ra = 10^{10}$.

$$8 \leq N_x \leq 16, \ 17 \leq N_y \leq 34, \text{ and } 40 \leq N_z \leq 80.$$
Performance at very high Rayleigh numbers

Meshes have been generated with the criteria of keeping the same number of points in the BL than for $Ra = 10^{10}$. 
The results illustrate the potential of $C_4$ regularization as a parameter-free turbulence model.

Robustnest. It preserves the symmetry properties and therefore, the solution cannot blow up even for very coarse meshes.

Test the performance of other forms of $C_4^\gamma$ regularization (with $\gamma \neq 1$).

Add some additional dissipation by (approximately) restoring the Galilean invariance.

$$(\partial_t)^\gamma_4 u_\epsilon = \partial_t(u_\epsilon - 1/2(1 + \gamma)u_\epsilon'') = G_4^\gamma(\partial_t u_\epsilon),$$

Since $(G_4^\gamma)^{-1}(\phi) \approx 2\phi - G_4^\gamma(\phi) + \mathcal{O}(\epsilon^6)$, an energetically almost equivalent set of equations can be derived

$$\partial_t u_\epsilon + C_4^\gamma(u_\epsilon, u_\epsilon) = D_4^\gamma u_\epsilon - \nabla p_\epsilon,$$

where $D_4^\gamma u = D u + 1/2(1 + \gamma)(Du')'$. 
Thank you for you attention
Further reading about $C_4$ regularization

