Spectrally-consistent regularization modelling of wind farm boundary layers

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Contents

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2. Regularization
3. Hyperviscosity
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DNS of turbulent incompressible flows

Main features of the DNS code:

- Pseudo-spectral method
- 3/2 rule de-aliasing technique
- Structured non-staggered grids
- Fully-explicit second-order time-integration method
- MPI parallelization

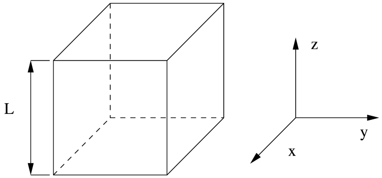
Offshore wind farm

Towards wind farm simulation

Steps to final DNS:

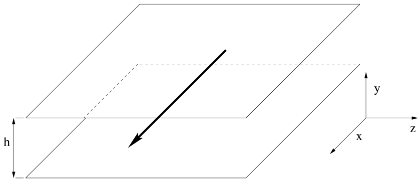
- Forced homogeneus isotropic turbulence
- Channel flow
- Atmospheric boundary layer
- Wind farm simulation

Forced homogoneus isotropic turbulent square box up to $Re_\lambda \approx 202$



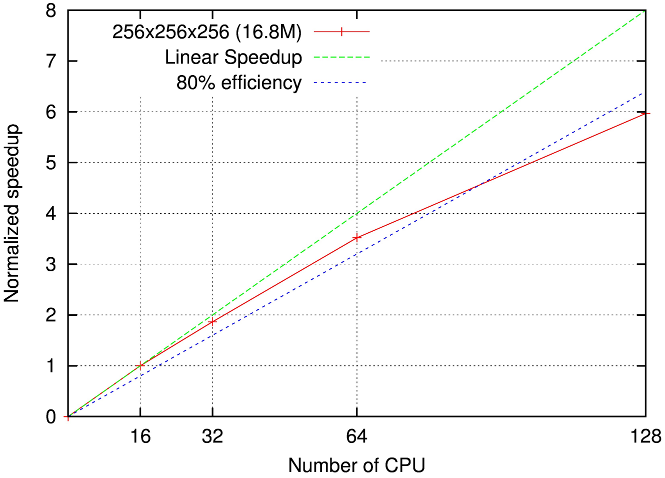
- Forced energy spectrum
- $L = 2\pi$
- $Re_\lambda \approx 72$, 202
- $\nu \approx 0.003, 0.0004$
- E_1, E_2 fixed energies

Channel flow at $Re_\tau = 180$



- $h = 2\delta = 2.0$
- $Re_\tau = 180$
- $U_0 = 1.0$
- $Re_0 = 3300$

Strong Scaling



Governing equations

Incompressible Navier-Stokes equations:

$$\begin{aligned} \nabla \cdot u &= 0 \\ \partial_t u + \mathcal{C}(u, u) &= \mathcal{D}u - \nabla p \end{aligned}$$

where the **nonlinear convective** term is given by

$$\mathcal{C}(u, \phi) = (u \cdot \nabla)\phi$$

and the **linear dissipative** term is given by

$$\mathcal{D}\phi = \nu \Delta \phi$$

Symmetry-preserving regularization modeling

We consider smooth approximations (regularizations) of the nonlinearity,

$$\partial_t u_\epsilon + \widetilde{\mathcal{C}}(u_\epsilon, u_\epsilon) \quad = \quad \mathcal{D}(u_\epsilon) - \nabla p_\epsilon$$

that conserve the following inviscid invariants

- Kinetic energy : (u, u)
- Enstrophy (in 2D) : (ω, ω)
- Helicity (in 3D) : (ω, u)

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The **approximate convective operator** must **preserve** the basic **symmetry** properties:

$$(\mathcal{C}(u, v), \; w) = -(\mathcal{C}(u, \; w), v)$$

$$(\mathcal{C}(u, v), \Delta v) = \quad (\mathcal{C}(u, \Delta v), v) \qquad \text{in 2D}$$

Symmetry-preserving regularization models

Finally

$$\boxed{\mathcal{C}_4^{\gamma}(u,v) = \frac{1}{2}\left((\mathcal{C}_4 + \mathcal{C}_6) + \gamma(\mathcal{C}_4 - \mathcal{C}_6)\right)(u,v)}$$

where $\mathcal{C}_4$ and $\mathcal{C}_6$ read

$$\mathcal{C}_4(u,v) = \mathcal{C}(\bar{u},\bar{v}) + \overline{\mathcal{C}(\bar{u},v')} + \overline{\mathcal{C}(u',\bar{v})}$$
$$\mathcal{C}_6(u,v) = \mathcal{C}(\bar{u},\bar{v}) + \mathcal{C}(\bar{u},v') + \mathcal{C}(u',\bar{v}) + \overline{\mathcal{C}(u',v')}$$

Symmetry-preserving regularization models

Taking $\gamma = 1$ we obtain the $\mathcal{C}_4$ approximation[0],

$$\partial_t u_\epsilon + \mathcal{C}_4(u_\epsilon, u_\epsilon) = \mathcal{D}(u_\epsilon) - \nabla p_\epsilon$$

in which the convective term in smoothed according to:

$$\boxed{\mathcal{C}_4(u, v) = \mathcal{C}(\bar{u}, \bar{v}) + \overline{\mathcal{C}(\bar{u}, v')} + \overline{\mathcal{C}(u', \bar{v})}}$$

where $u' = u - \bar{u}$ and $\mathcal{C}_4(u, v) = \mathcal{C}(u, v) + \mathcal{O}(\epsilon^4)$ for **any symmetric filter**.

[0] Roel Verstappen, **Computers & Fluids**, 37 (7): 887-897, 2008

Symmetry-preserving regularization models

Two main drawbacks:

- Additional hump in the tail of the energy spectrum
- For very coarse meshes, the damping factor can take very small values

How do we adress it?

Symmetry-preserving regularization models

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- Additional hump in the tail of the energy spectrum
- For very coarse meshes, the damping factor can take very small values

How do we adress it?

Answer: Restoring the galilean invariance with an hyperviscosity effect.

Restoring the Galilean invariance: hyperviscosity effect

The $\mathcal{C}_4^\gamma$ regularization

$$\partial_t u_\epsilon + \mathcal{C}_4^\gamma(u_\epsilon, u_\epsilon) = \mathcal{D}(u_\epsilon) - \nabla p_\epsilon \;\; ; \qquad \nabla \cdot u_\epsilon = 0$$

preserves all the invariant transformations of the original NS equations, **except the Galilean transformation**. To restore it, the time-derivative, $\partial_t u_\epsilon$, needs to be replaced by the following fourth-order approximation:

$$(\partial_t)_4^\gamma u_\epsilon = \partial_t(u_\epsilon - 1/2(1+\gamma)u_\epsilon'') = \mathcal{G}_4^\gamma(\partial_t u_\epsilon),$$

Since $(\mathcal{G}_4^\gamma)^{-1}(\phi) \approx 2\phi - \mathcal{G}_4^\gamma(\phi) + \mathcal{O}(\epsilon^6)$, an energetically almost equivalent set of equations can be derived:

Restoring the Galilean invariance: hyperviscosity effect

$$\boxed{\partial_t u_\epsilon + \mathcal{C}_4^\gamma(u_\epsilon, u_\epsilon) = \mathcal{D}_4^\gamma u_\epsilon - \nabla p_\epsilon \quad ; \qquad \nabla \cdot u_\epsilon = 0}$$

Compared with the $\mathcal{C}_4^\gamma$ regularization, the $\{\mathcal{CD}\}_4^\gamma$ equations reinforce the dissipation by means of the hyperviscosity term

$$\mathcal{D}_4^\gamma u = \mathcal{D}u + 1/2(1+\gamma)(\mathcal{D}u')'$$

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Recalling that $u' = -(\epsilon^2/24)\Delta u + \mathcal{O}(\epsilon^4)$, the additional dissipation to the kinetic energy is approximately given by

$$\varepsilon^{reg} \approx -\frac{1}{2}(1+\gamma)\left(\frac{\epsilon^2}{24}\right)^2 \left(\Delta u_\epsilon, \Delta^2 u_\epsilon\right)$$

Restoring the Galilean invariance: hyperviscosity effect

The spectral representation is

$$\mathcal{C}_4^{\gamma}(u,v)_k = i\Pi(k) \sum_{p+q=k} f_4^{\gamma}\left(\widehat{g}_k, \widehat{g}_p, \widehat{g}_q\right) \hat{u}_p q \hat{v}_q$$

$$\mathcal{D}_4^{\gamma} u_k = h_4^{\gamma}(\widehat{g}_k)\nu|k|^2 \hat{u}_k$$

so the diffusive term is multiplied by

$$h_4^{\gamma}(\widehat{g}_k) = 1 + \tilde{\gamma}(1-\widehat{g}_k)^2$$

where $h_4 \geq 1$.

Stopping the vortex-stretching[1]

Taking the curl of momentum equation the **vorticity transport equation** follows

$$\partial_t \omega + \mathcal{C}\left(u, \omega\right) = \color{blue}{\mathcal{C}\left(\omega, u\right)} + \color{red}{\mathcal{D}(\omega)}$$

[1] F.X. Trias *et al.* **Computers&Fluids**, 39:1815-1831, 2010

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Let us now consider **periodic boundary conditions**. Then, the **enstrophy equation**:

$$\frac{1}{2}\frac{d}{dt}(\omega, \omega) = (\omega, \mathcal{C}(\omega, u)) - \nu\left(\nabla\omega, \nabla\omega\right)$$

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$$\frac{1}{2}\frac{d}{dt}(\omega, \omega) = (\omega, \mathcal{C}(\omega, u)) - \nu(\nabla\omega, \nabla\omega)$$

Unless the grid is fine enough, convection dominates diffusion:

$$(\omega, \mathcal{C}(\omega, u)) > \nu(\nabla\omega, \nabla\omega)$$

[1] F.X. Trias *et al.* **Computers&Fluids**, 39:1815-1831, 2010

Stopping the vortex-stretching

In order to prevent local intensification of vorticity, dissipation must dominate the vortex-stretching term at the smallest grid scale.

$$\frac{\hat{\omega}_{k_c} \cdot \mathcal{C}_4^{\gamma}(\omega, u)^*_{k_c} + \mathcal{C}_4^{\gamma}(\omega, u)_{k_c} \cdot \hat{\omega}^*_{k_c}}{2\hat{\omega}_{k_c} \cdot \hat{\omega}^*_{k_c}} \leq h_4^{\gamma}\left(\hat{g}_k\right) \nu k_c^2$$

Very difficult to compute...

Stopping the vortex-stretching

We want that the triadic interactions at the smallest scale be independent of the interacting pairs

$$f_4^{\gamma}(\widehat{g}_{k_c}, \widehat{g}_p, \widehat{g}_q) \approx f_4^{\gamma}(\widehat{g}_{k_c}).$$

So the overall damping effect at the smallest grid scale is

$$H_4(\widehat{g}_{k_c}) = f_4^{\gamma}(\widehat{g}_{k_c}) / h_4^{\gamma}(\widehat{g}_{k_c})$$

with the crucial condition that $0 < H_4(\widehat{g}_{k_c}) \leq 1$.
Then...

Stopping the vortex-stretching

Then... we have a regularization model with two parameters:

- overall damping $H_4(\widehat{g}_{k_c})$.
- gamma γ

How can we deal with them?

On the determination of $H_4(\widehat{g}_{k_c})$

We can express $H_4(\widehat{g}_{k_c})$, as a function of the invariants R and Q of the strain tensor, $S(u) = 1/2(\nabla u + \nabla u^T)$.
where

$$R = -1/3tr(S^3) = -det(S) = -\lambda_1\lambda_2\lambda_3$$

$$Q = -1/2tr(S^2) = -1/2(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)$$

On the determination of $H_4(\widehat{g}_{k_c})$

The diffusive term can be bounded by its largest eigenvalue

$$(\nabla\omega, \nabla\omega) = -(\omega, \Delta\omega) \leq -\lambda_\Delta (\omega, \omega),$$

Finally

$$\boxed{H_4(\widehat{g}_{k_c}) = \min\left\{\nu\lambda_\Delta \frac{Q}{R^+}, 1\right\}}$$

where $R^+ = max\{R, 0\}$

On the determination of $H_4(\widehat{g}_{k_c})$

Main features of the model:

- Lower bound for $H_4(\widehat{g}_{k_c})$
- It automatically switches off when h approaches to the smallest scale in a turbulent flow
- It automatically switches off for laminar flows (no vortex-stretching) and 2D flows
- It automatically switches off in the wall

On the determination of γ

- We assume that the smallest grid scale $k_c = \pi/h$ lies within the inertial range for a classical Kolmogorov energy spectrum

$$E(k) = C_K \varepsilon^{2/3} k^{-5/3}$$

The total dissipation for $k_T \leq k \leq k_c$ can be approximated by the contribution of the following two terms

$$\mathcal{D}_\nu \equiv \nu \int_{k_T}^{k_c} k^2 E(k) dk,$$

$$\mathcal{D}''_\nu \equiv \nu \int_{k_T}^{k_c} k^4 \alpha^4 E(k) dk,$$

where $\mathcal{D}_\nu$ is the physical viscous dissipation and $\mathcal{D}''_\nu$ is the additional dissipation introduced by the hyperviscosity term, $(\mathcal{D}u')'$.

On the determination of γ

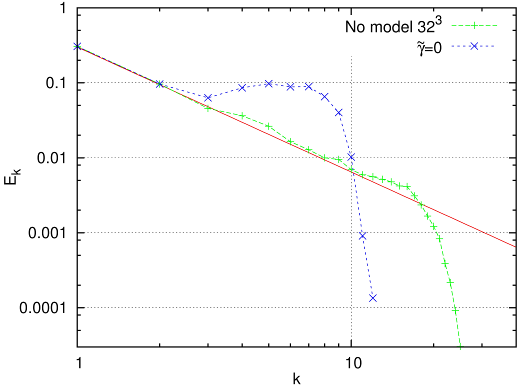
- We assume that $\tilde{H}_4 = \mathcal{O}(H_4(\widehat{g}_{k_c}))$
- We can bound the ratio R^+/Q by Q and hence $\tilde{H}_4$
- We can compute approximately the invariant Q with Kolmogorov spectrum

Then

$$\tilde{\gamma} \gtrsim 4\left\{8C_K^{-3/2} - \left(1 - \left(\frac{k_T}{k_c}\right)^{4/3}\right)\right\} \approx 4\left(8C_K^{-3/2} - 1\right)$$

Hence, for a Kolmogorov constant of $C_K \approx 1.58$ it leads to a lower limit of $\tilde{\gamma} \approx 12.1$ or ($\gamma \approx 23.2$)

Test-case: Forced homogeneus isotropic turbulence



Energy spectra at $Re_\lambda \approx 72$ for different values of $\tilde{\gamma}$ from 0 up to 30

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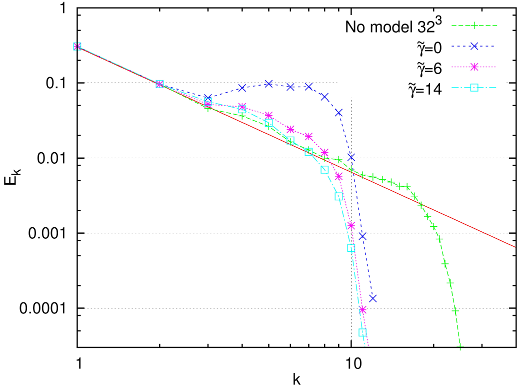

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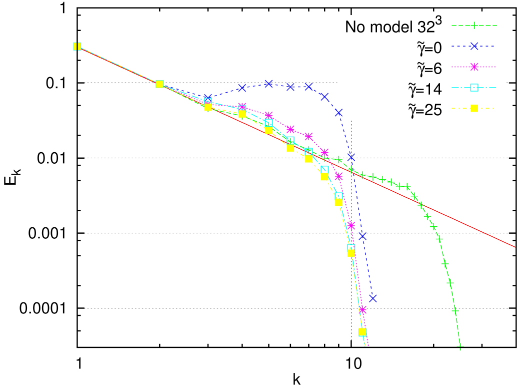

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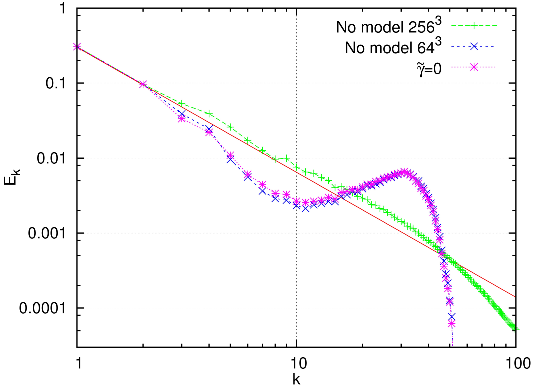
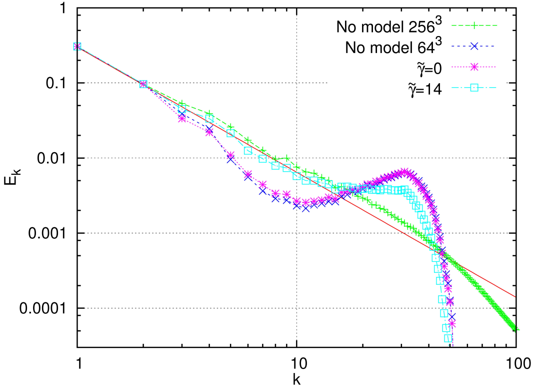


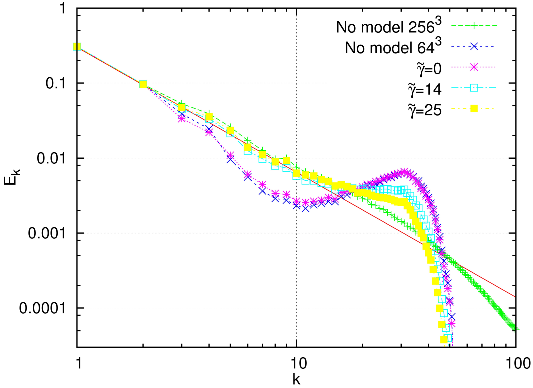
Energy spectra at $Re_\lambda \approx 202$ for different values of $\tilde{\gamma}$ from 0 up to 30

Test-case: Forced homogeneus isotropic turbulence



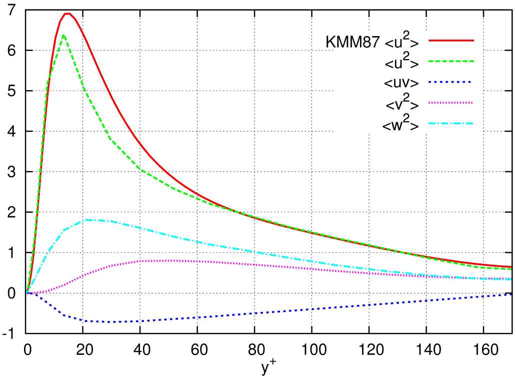
Energy spectra at $Re_\lambda \approx 202$ for different values of $\tilde{\gamma}$ from 0 up to 30

Test-case: Forced homogeneus isotropic turbulence



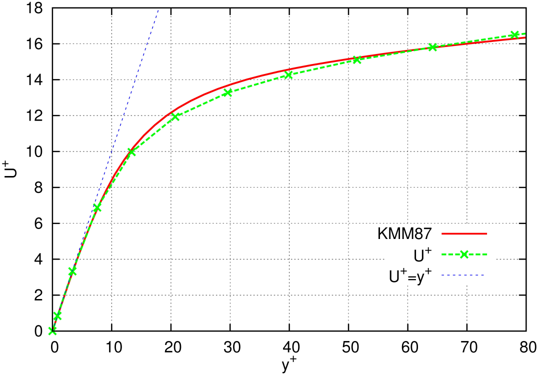
Energy spectra at $Re_\lambda \approx 202$ for different values of $\tilde{\gamma}$ from 0 up to 30

Channel flow $Re_\tau = 180$



Grid size 32x32x32. Some turbulent statistics

Channel flow $Re_\tau = 180$



Grid size 32x32x32. Mean streamwise velocity profile

Conclusions

- The numerical results illustrate the potential of $\{\mathcal{CD}\}_4^\gamma$ regularization as a **parameter-free turbulence model**.

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- The numerical results illustrate the potential of $\{\mathcal{CD}\}_4^\gamma$ regularization as a **parameter-free turbulence model**.
- **Robustnest**. It preserves the symmetry properties and therefore, the solution cannot blow up even for very coarse meshes.

Conclusions and Future Research

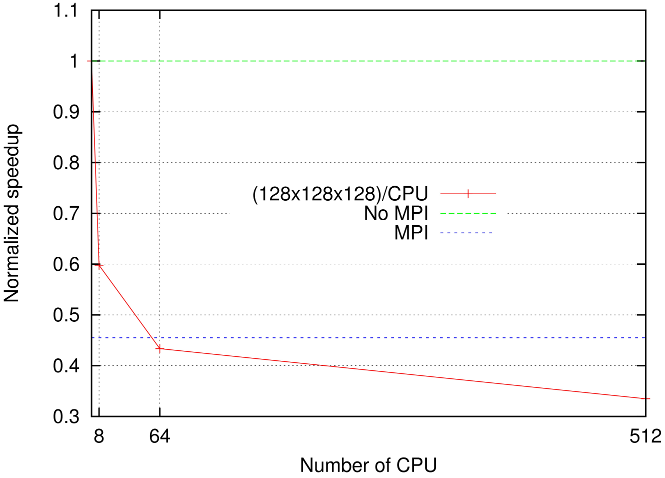
- $\{\mathcal{CD}\}_4^\gamma$ regularization of channel flow
- **Wind farm simulation**

Thank you for you attention

Further reading about $\mathcal{C}_4$ regularization

- Roel Verstappen, *"On restraining the production of small scales of motion in a turbulent channel flow"*, Computers & Fluids, 37 (7): 887-897, 2008
- F. X. Trias *et al.*, *"Parameter-free symmetry-preserving regularization modeling of a turbulent differentially heated cavity"*, Computers & Fluids, 39:1815-1831, 2010.
- F. X. Trias and R.W.C.P. Verstappen, *"On the construction of discrete filters for symmetry-preserving regularization models"*, Computers & Fluids, 40:139-148, 2011.

Weak Scaling



On the determination of γ

Hence

$$\mathcal{D}_\nu + \tilde{\gamma}\mathcal{D}''_\nu = \frac{3\nu}{16} C_K \varepsilon^{2/3} \left\{ \left(4 + \tilde{\gamma}\alpha^4 k_c^4\right) k_c^{4/3} - \left(4 + \tilde{\gamma}\alpha^4 k_T^4\right) k_T^{4/3} \right\}$$

where $\tilde{\gamma} = 1/2(1+\gamma)$. At the tail of the spectrum

$$\tilde{H}_4 \approx \frac{\mathcal{D}_\nu + \tilde{\gamma}\mathcal{D}''_\nu}{\varepsilon}$$

represents the ratio between the total dissipation and the energy transferred from scales larger than k_T