Abstract

Direct numerical simulations of the incompressible Navier-Stokes equations are limited to relatively low-Reynolds numbers. Therefore, dynamically less complex mathematical formulations are necessary for coarse-grain simulations. Regularization and eddy-viscosity models for LES are examples thereof. They rely on differential operators that should capture well different flow configurations (laminar and 2D flows, near-wall behavior, transitional regime...). Most of them are based on the combination of invariants of a symmetric second-order tensor that is derived from the gradient of the resolved velocity field. In the present work, they are presented in a framework where all the models are represented as a combination of elements of a 5D phase space of invariants. In this way, new models can be constructed by imposing appropriate restrictions in this space.

DNS of turbulent incompressible flows

Turbulent square duct at \( Re = 1200 \) (127M grid points)

Air-filled differentially heated cavity at \( Ra = 10^4 \) (111M grid points)

Square cylinder at \( Re = 22000 \) (300M grid points)

Finding proper bounds for the model (constants?)

Then, the following three SStx models for LES can be considered

\[ \nu_{SStx}^{2} = \left( C_{F} \right)^{2} \left( \frac{R_{0}}{R_{s}} \right)^{2} \left( \frac{Q_{a}}{Q} \right)^{2} \]

The model constants, \( C_{Stx} \), can be related with the Vreman’s constant, \( C_{Vr} \), with the following inequality

\[ 0 \leq \left( \frac{C_{Stx}}{C_{Vr}} \right)^{2} \leq 1 / \frac{3}{3} \]

Hence, imposing that \( C_{Stx} = C_{Stx} = C_{Stx} = \sqrt{3} C_{Stx} \) guarantees:

- numerical stability \( 0 \leq \nu_{Stx}^{2} \leq \nu_{Stx}^{2} \)
- less or equal dissipation than Vreman’s!

Theory: a 5D phase space for eddy-viscosity models

Many turbulence eddy-viscosity models for LES have been proposed in the last decades...

\[ \partial_{t} \mathbf{U} + (\mathbf{U} \cdot \nabla) \mathbf{U} = -\nabla \mathbf{P} - \nabla \cdot \mathbf{r} \mathbf{U} + \nabla \cdot \mathbf{S} \]

...most of them rely on differential operators that are based on the combination of invariants of a symmetric second-order tensor derived from \( \mathbf{G} = \mathbf{S} / \partial_{t} \). It can be characterized by 5 basic invariants

\[ \mathcal{Q}_{4} = \mathcal{Q}_{5} = \mathcal{Q}_{6} = \mathcal{Q}_{7} = \mathcal{Q}_{8} = \mathcal{Q}_{9} \]

...where \( \mathcal{Q}_{4} = 1/2(\mathbf{A} - \mathbf{A}^{T}) \) and \( \mathcal{Q}_{5} = \det(A) \) represent the second and third invariants of the second-order tensor \( \mathbf{A} \), \( \mathcal{Q}_{6} = \mathcal{Q}_{7} = \mathcal{Q}_{8} = \mathcal{Q}_{9} \) are the three singular eigenvalues of \( \mathbf{G} \), i.e., \( \mathcal{Q}_{1} = \sqrt{\lambda_{1}} \) where \( \mathcal{Q}_{1} \) is an eigenvalue of \( \mathbf{G}^{T} \), \( \lambda_{2} = \lambda_{3} = \frac{1}{2} \sqrt{\lambda_{1}^{2} - \lambda_{1}} \).

R\text{2}-based model

\[ \nu_{r2}^{2} = \left( C_{F} \right)^{2} \left( \mathcal{Q}_{5} / \mathcal{Q}_{4} \right) \]

Smagorinsky model

\[ \nu_{smag}^{2} = \left( C_{F} \right)^{2} \left( \mathcal{Q}_{5} / \mathcal{Q}_{4} \right)^{2} \]

WALE model

\[ \nu_{w}^{2} = \left( C_{F} \right)^{2} \left( -2 \mathcal{Q}_{5} / \mathcal{Q}_{4} + 2 \mathcal{Q}_{5} / \mathcal{Q}_{4} + 3 \mathcal{Q}_{5} / \mathcal{Q}_{4} \right) \]

Vreman’s model

\[ \nu_{v}^{2} = \left( C_{F} \right)^{2} \left( (1/2)(\mathcal{Q}_{5} - \mathcal{Q}_{6} + \mathcal{Q}_{7}) / (\mathcal{Q}_{5} - \mathcal{Q}_{8}) \right)^{2} \]

Sigma model

\[ \nu_{\sigma}^{2} = \left( C_{F} \right)^{2} \left( (1/2)(\mathcal{Q}_{5} - \mathcal{Q}_{6} + \mathcal{Q}_{7}) / (\mathcal{Q}_{5} - \math{Q}_{8}) \right)^{2} \]

Preliminary results for a turbulent channel flow at \( Re = 395 \)

Building new proper invariants for LES models

Hence, new models can be derived by imposing restrictions on the differential operators they are based on...

\[ \mathbf{G} \]

...for instance, let us consider models that are based on the invariants of the tensor \( \mathbf{G}^{T} \) (like Vreman’s and Sigma models)

\[ \nu = \left( C_{F} \right)^{2} \left( \mathcal{Q}_{5} / \mathcal{Q}_{4} \right) \]

\[ \nu = \left( C_{F} \right)^{2} \left( \mathcal{Q}_{5} / \mathcal{Q}_{4} \right) \]

Formula

\[ A \times \mathbf{G}^{T} \]

Wall behavior

\[ \mathbf{S} \]

Models

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<th>Vreman’s ( R_{2} )-based model</th>
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Further reading