A projected Ghost Fluid Method for a mimetic approach to extreme contrast interfaces in multiphase flows

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Overview

1 Introduction

2 Mimetic & Symmetry-Preserving Schemes

3 Multiphase flows
   - Interface capturing
   - Interface reconstruction

4 Discussion
Motivation

Goal
Develop numerical methods for the accurate computation of multiphase flows under extreme contrast interfaces

Application
Numerical study of heat and mass transfer in a LiBr falling film absorber for a solar absorption chiller

Challenges
- extreme interface density ratio
- instabilities at low Reynolds
- non-condensable gases reaching the interface
Mono-phase approach

Governing equations

\[
\begin{align*}
\frac{\partial \rho \vec{u}}{\partial t} + \nabla \cdot (\rho \vec{u} \otimes \vec{u}) &= \nabla \cdot \bm{S} \\
\nabla \cdot \vec{u} &= 0 \\
\frac{\partial e}{\partial t} + \nabla \cdot (e \otimes \vec{u}) &= \nabla \cdot \lambda \nabla T
\end{align*}
\]

\[
\begin{align*}
\bm{S} &= \mu \bm{E} - \nabla P \\
\bm{E} &= \frac{\nabla u + \nabla u^T}{2}
\end{align*}
\]
Multi-phase approach

Jump conditions

\[
\begin{align*}
[\rho]_T &= \rho_I - \rho_V \\
[\mu]_T &= \mu_I - \mu_V \\
[\lambda]_T &= \lambda_I - \lambda_V \\
[\vec{u}]_T \cdot \hat{n}_T &= -\dot{m} \left[ \frac{1}{\rho} \right]_T \\
[S]_T \cdot \hat{n}_T &= \sigma \kappa \hat{n}_T \\
[\nabla T]_T \cdot \hat{n}_T &= L_{vap} \dot{m}
\end{align*}
\]

Research question

*Can we impose the jump conditions above in a **sharp** and **physically consistent** way?*
Keep all your energy

Symmetry-preserving

$$\frac{d}{dt}(u_f^* \Omega_c u_f) = - u_f^* (C(u_f) + C(u_f)^*) u_f$$
$$- u_f^* (D + D^*) u_f$$
$$- u_f^* (G_{pc}) (G_{pc})^* u_f = - u_f^* (D + D^*) u_f$$

$$(\nabla p, \bar{u}) = -(p, \nabla \cdot \bar{u}) \rightarrow (G_{pc})^* \Omega_f u_h = - p_c^* \Omega_c Mu_f$$

... a symmetry-preserving, spatial discretization of the Navier–Stokes equations is unconditionally stable and conservative.

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Metric is all you need

Mimetic method\(^2\)

\[
\begin{align*}
(\vec{u}, \nabla p) &= -(\nabla \cdot \vec{u}, p) \quad \rightarrow \quad (u_f, Gp_c) = -(Mu_f, p_c) \\
(v, w) &= \int_{\Omega} vw \quad \rightarrow \quad (v_h, w_h) = v_h^* \Omega_h w_h
\end{align*}
\]

\[
\int_{\Omega} \nabla \cdot \vec{u} = \int_{\partial \Omega} \vec{u} \hat{n}_f \approx \sum_f u_f S_f
\]

The construction of the derived operators is based on the duality principle . . .

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A graph to rule them all

Divergence/primary operator

\[ \int_{\Omega} \nabla \cdot \tilde{u} = \int_{\partial \Omega} \tilde{u} \hat{n}_f \approx T_{cf} S_f u_f \]

\[
T_{cf} = \begin{pmatrix}
    f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 & f_8 & f_9 \\
    c_1 & 0 & 0 & -1 & +1 & +1 & 0 & 0 & 0 & 0 \\
    c_2 & -1 & -1 & +1 & 0 & 0 & 0 & 0 & 0 & 0 \\
    c_3 & +1 & 0 & 0 & 0 & 0 & -1 & +1 & 0 & 0 \\
    c_4 & 0 & +1 & 0 & 0 & 0 & 0 & 0 & 0 & +1 & +1
\end{pmatrix}
\]
Summary

**Metric**

\[ \Omega_f = -\Delta_x S_f \]

**Symmetry-preserving**

\[ G = -\Omega_f^{-1} M^* = -(\Delta_x S_f)^{-1} (T_{cf} S_f)^* = -(\Delta_x)^{-1} T_{cf}^* \]

**Mimetic**

\[ (u_h, G p_c) = -(M u_h, p_c) = u_h^* \Omega_f G p_c = u_h^* M^* p_c \]

**Highlight**

The definition of the **metric** of the dual space induces the symmetry-preserving/mimetic **dual operator**
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Level-Set method

**Definition**

Level-Set (signed) function:

\[ r_c = (x_{\Gamma} - x_c) \cdot \hat{n}_{\Gamma} \]

**Algorithm**

\[ \psi_c = \frac{1}{2} \left( \tanh \left( \frac{r_c}{2\epsilon} \right) + 1 \right) \]

Advection:

\[ \frac{\partial \psi_c}{\partial t} + \nabla \cdot (\psi_c \otimes \vec{u}) = 0 \]

Recompression:

\[ \frac{\psi c}{\tau \rightarrow \infty} + \nabla \cdot \left( \psi_c (1 - \psi_c) \right) = \epsilon \nabla^2 \psi_c \]

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Ghost Fluid Method

Definition

\[ \nabla \cdot \lambda \nabla u = f \]
\[ [u]_\Gamma = a \]
\[ [\lambda \nabla u]_\Gamma \cdot \hat{n}_\Gamma = b \]

Algorithm\(^4\)

\[ (\lambda \nabla u)^\pm = (r_c^{\pm1} T_{pi} - u_\Gamma)^\pm \]
\[ [u]_\Gamma = a \]
\[ [\lambda \nabla u]_\Gamma \cdot \hat{n}_\Gamma = b \]

Solve for \((\lambda \nabla u)^\pm\)

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Ghost Fluid Method

Final forms

\[ \nabla \cdot \lambda \nabla u \approx M \hat{\Lambda}_f G u_f - M \hat{\Lambda}_f (\Delta_x)^{-1} Q a_f \]
\[ - \Lambda_f \Omega_c^{-1} H^* \cos(\Theta) S_f (\Lambda_f^\pm)^{-1} b_f \]

\[ \lambda \nabla u \approx \hat{\Lambda}_f G u_f - \hat{\Lambda}_f (\Delta_x)^{-1} Q a_f \]
\[ - (\Lambda_f)^{-1} (\Delta_x)^{-1} (R^\pm)^{-1} \Lambda_f^\pm b_f \]

Where:

- \( Q \): interface orientation
- \( \hat{\Lambda}_f \): harmonic mean of \( \lambda \)
Ghost Fluid Method

Research question

Does this discretization lie in the image of $L$?

Static phase change

In a static phase change situation, provided Neumann boundary conditions, is the discretization consistent? Condition:

\[
\int_{\Omega} \nabla \cdot \lambda \nabla u = \int_{\Omega} f = \int_{\Gamma} [\lambda \nabla u]_\Gamma \cdot \hat{n}_\Gamma \quad (1)
\]

However:

\[
(1, f_h + M \hat{\Lambda}_f (\Delta_x)^{-1} Qa_f + \Lambda_f \Omega_c^{-1} H^* \cos(\Theta) S_f (\Lambda_f^\pm)^{-1} b_f ) \neq 0
\]

Which is not satisfied, unfortunately.
Ghost Fluid Method

Problem

High coefficient ratios magnify this problem! This may lead to divergence!

Figure 1: Ghost Fluid Method at $\lambda_+ / \lambda_- = 1$.

Figure 2: Ghost Fluid Method at $\lambda_+ / \lambda_- = 1E4$. 
Remedy

Force the system \( L \vec{u} = \vec{s} \) to have a solution by projecting over the image space.

Evenly share the imbalance between all cells.

Remark

Force the \( \vec{s} \in \text{img}(L) \) by:

\[
\vec{s} = \vec{s} - \frac{\langle \vec{1}_c, \vec{s} \rangle}{|\vec{1}_c|} \vec{1}_c \tag{2}
\]

Figure 3: Project \( \vec{s} \) over \( \text{img}(A) \).
Projected Ghost Fluid Method

**Remedy**

Force the system $L\tilde{u} = \tilde{s}$ to have a solution by projecting over the image space.

**Remark**

Force the $\tilde{s} \in \text{img}(L)$ by:

$$\tilde{s} = \tilde{s} - \frac{\langle \mathbf{1}_c, \tilde{s} \rangle}{|\mathbf{1}_a|} \mathbf{1}_a \quad (3)$$

Share the imbalance between adjacent cells.

*Figure 4: Project $\tilde{s}$ over $\text{img}(L)$.***
Projected Ghost Fluid Method

Remedy

Force the system to have a solution by projecting over the image space.

Figure 5: PGFM at $\lambda_+ / \lambda_- = 1E4$.

Figure 6: LPGFM at $\lambda_+ / \lambda_- = 1E4$. 
Projected Ghost Fluid Method

Figure 7: Error norm vs $\lambda_+/\lambda_-$.  
Figure 8: Convergence order vs $\lambda_+/\lambda_-$. 
Challenges and opportunities

**Pros**
- sharp method
- explicit treatment of discontinuities

**Cons**
- lack of geometric information at the interface
- flow jumps are not in the image of $L$

**Opportunities**
- weighted projection
- equivalent interface reconstruction
Thank you for your attention
Research question
Does this discretization preserve duality?

Bubble flow
Is the kinetic energy variation due to the pressure jump properly captured by the Ghost Fluid Method?

\[(u_f, G p_c - (\Delta x)^{-1} Q a_f) + (M u_f, p_c) = -u_f^* S_f Q a_f\]

Which is an approximation of the balance between kinetic and surface potential energy.
Warning!

Because \((\lambda \nabla u)^\pm\) is defined at the interface, we need to move it back to the face to operate with a regular divergence operator. This is required to move from a differential approach to an integral one.

Tangency condition

\[ r_{c+} - r_{c-} = \hat{n}_\Gamma \cdot (\vec{x}_{c+} - \vec{x}_{c-}) \]

Homothecy condition

\[ \vec{x}_h = \frac{r_{c+}}{r_{c+} - r_{c-}} \vec{x}_{c-} - \frac{r_{c-}}{r_{c+} - r_{c-}} \vec{x}_{c+} = H(r_c)\vec{x}_c \]
Ghost Fluid Method

Geometrical Warning!

Interface is not uniquely defined!
This prevents us from properly defining the interface metric.

Side Note

$$\Omega_i = S_f |\vec{x}_h - \vec{x}_{c+}|$$

If the following holds:  
- cell and face centroids are aligned
- plane does not intersect the base

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